The role of jumps and options in the risk premia of interest rates

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Abstract

There is evidence that jumps double the explanation power of Campbell and Shiller (1991) excess bond returns regressions (Wright and Zhou, 2009), and options bring information about bond risk premia beyond that spanned by the yield curve (Joslin, 2007). In this paper I incorporate these features in a Gaussian Affine Term Structure Model (ATSM) in order to assess two questions: (1) what are the implications of incorporating jumps in an ATSM for option pricing, and (2) how jumps and options affect the bond risk-premia dynamics. The main findings are: (1) jump risk-premia is negative in a scenario of decreasing interest rates and explain 10%-20% of the level of yields, (2) options help to reconcile, in part, the weak form of Expectation Hypothesis, and (3) gaussian models without jumps and with constant intensity jumps are good to price options.

1 Introduction

Understanding the behavior of the term structure of interest rates is important for both practitioners and policy makers. The first want to predict its behavior in order to undertake profitable positions in bonds and interest rate derivatives and better assess interest rate risks, while the second want to extract its economic content to provide policy decisions.

There is empirical evidence that both jumps and interest rate options are important to describe the risk premia behavior of interest rate. For instance, Wright and Zhou (2009) show that by adding a measure of market jump volatility risk as an explanatory variable in the excess bond returns regressions (a la Campbell and Shiller (1991)) double their R^2s. Their result suggests that the addition of jump processes in a term-structure model might improve the understanding of the yield curve future behavior and its risk premia. In fact, Ludvigson and Ng (2009) find that real and inflation factors have important forecasting power for future excess returns in bonds beyond the one contained in the term structure. Jiang and Yan (2006) and Johannes (2004) show that jumps in interest rates are related to geopolitical events and surprises in macroeconomics releases. These results taken together suggest that jumps are somehow connected to the macroeconomics. In addition they indicate that the incorporation of jumps that have their risks priced can make Term-Structure models more suitable to analyze risk-premia dynamics and to evaluate option pricing.

On the other hand, Joslin (2007), Almeida et al. (2006) and Graveline (2007) show that options can bring information about risk premia beyond that spanned by interest rates only. In fact, a large number of papers include interest rate options in the estimation to better price risks in Term Structure Models (Almeida et al. (2006), Almeida and Vicente (2009b), Almeida and Vicente (2009a), Joslin (2007) and Graveline (2007)). They say that the use of options helps to better identify the market price of risk dynamics. Indeed, if options have information beyond that within the yield curve, their inclusion will help to estimate and capture state variables dynamics that drive future bond excess returns that are not spanned by the yield curve.

The aim of this paper is to put these two features together in an Affine Term Structure Model (ATSM) in order to assess two questions: (1) what are the implications of incorporating jumps in an ATSM for option pricing, and (2) how jumps and options affect the bond risk-premia dynamics.

In this work I estimate six ATSMs with three factors: (1) Gaussian (A30), (2) Gaussian with options (A30o), (3) Gaussian with random jumps and constant intensity (A30J), (4) Gaussian with random jumps, constant intensity and options (A30oJ), (5) Gaussian with random jumps and time-varying intensity (A30JT), and (6) Gaussian with random jumps and time-varying intensity (A30oJT). My goal is to investigate model implied option price, as well as the risk premium behavior, particularly, jump premium[1].

The main findings are: (1) jump risk-premia is negative in a scenario of decreasing interest rates, and has significant average magnitude of 1%-2%, i.e., explains 10% to 20% of the level of yields; and (2) gaussian models and gaussian models with constant intensity jumps are better to price options.

Some related papers are Pan (2002), Jiang and Yan (2006), and Joslin (2007). The first uses an Affine Jump Diffusion (AJD) Model to examine the joint time series of the $S\&P500$ index and its near-the-money short dated option. Its model is able to capture risk premia coming from both stochastic volatility and jumps. The distinguishing feature of my work is modelling jumps for fixed income instruments.

[1] The premium here is defined as the difference between the risk neutral measure and the physical measure.

The second develops AJD and Quadratic Jumps Diffusion models for the Term-Structure to investigate volatility risk premia. But, it does not include option data in its estimation and it says nothing about the jump risk premium of interest rate levels. The last develops weakly incomplete affine models for the term-structure to show that the inclusion of options in the estimation can bring information about volatility risk premia, and moreover a large fraction of this risk can be hedged using only bonds. But, it is silent about jump risk premia.

2 Data Description

I use daily Brazilian Term Structure and Asian Interest Rate Option data ranging from $2-Jan-2006$ to $23-Nov-07$. The term structure is constructed from the DI future contracts traded at BM&F (the Brazilian Futures and Commodities Exchange House, similar to CBOT in Chicago, US). I also use the DI interest rate from the Interbank loan market of one day (similar to the LIBOR rate), whose contracts are registered at CETIP, to pin down the short-term of the yield curve. I interpolate the interest rate of all available maturities in each sample day to construct a constant-maturity interest rate dataset of 1, 21, 42, 63, 126, 189, 252, 378 and 504 working days (wd). The graph at the top-left of figure 1 shows evolution of the interpolated interest rates. And, the figure 2 plots the Interbank DI interest rate. Note that it is practically a pure jump process.

The Interbank DI interest rate gives rise to the DI Index, that is constructed by accruing it every day

$$IDI_T = \prod_{i=0}^{T-t-1} IDI_t(1 + DI_{t+i,t+i+1})^{1/252} \tag{1}$$

The Option data that I use has the DI Index as the object. The contracts are registered at BM&F, that gently gave me its dataset of Black-implied volatilities. I constructed a constant moneyness-maturity volatility dataset by interpolating the volatilities of all available options. The synthetic options have moneyness of 0.99, 1 and 1.01, and maturity of 240 working days. Here, Moneyness is defined as

$$Moneyness \triangleq \frac{VP(Strike)}{Price} \tag{2}$$

Some other studies, as Duffie et al. (2000), define moneyness as the strike-price ratio, but my definition is more precise. For call (sell) options, if the moneyness is equal to one the option is at-the-money, less than one is in-the-money (out-the-money), and greater than one is out-the-money (in-the-money). The graphs at the top-right and in the bottom of figure 1 show the interpolated volatilities for in, at

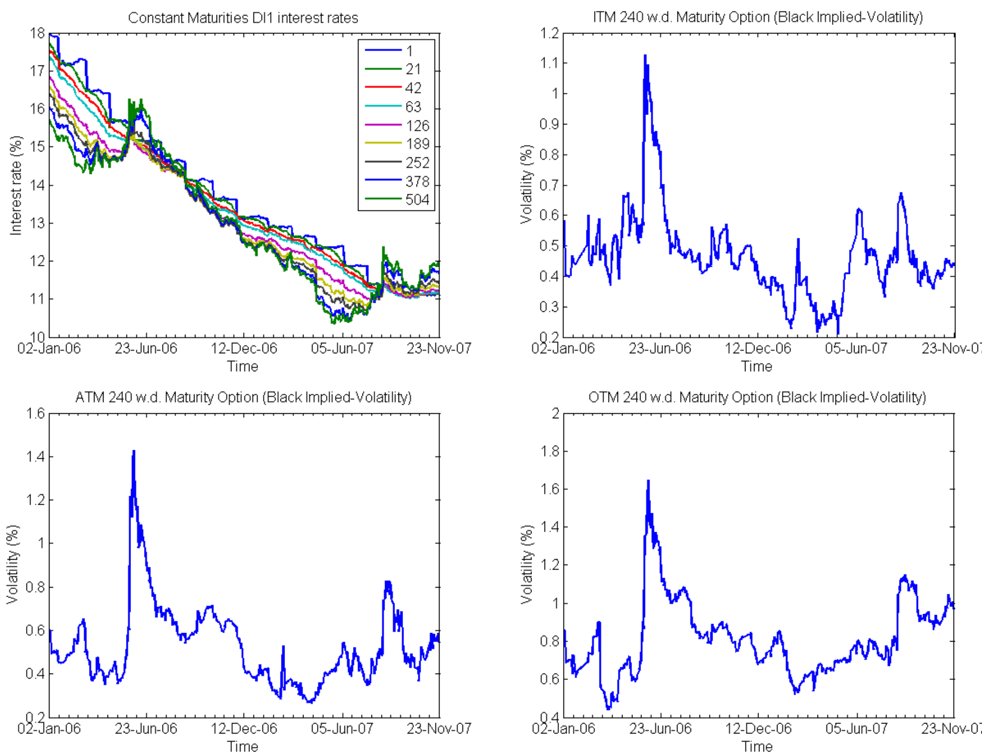

Figure 1: The daily evolution of the term structure of interest rate and of the volatility smirk for options with maturitiy 240 working days and moneyness of $(0.99, 1, 1.01)$.

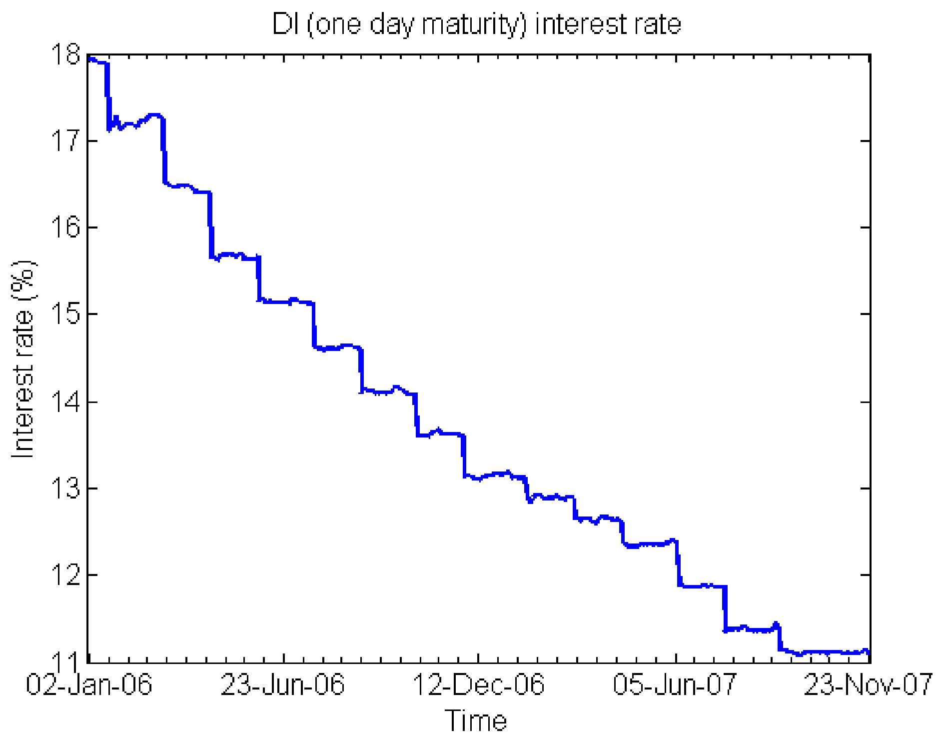

Figure 2: One day maturity Brazilian DI interest rate (annualized).

and out-the-money options of maturity 240 working days. Note how the volatility decreases with call option moneyness, i.e., when going from the out-the-money to the in-the-money options.

I estimate several models using either joint option-term structure data or only yield curve data. For the ones that option was used, I only included at and out-the-money options, because the in-the-money one seemed to have liquidity problems. As the models don't account for liquidity premia, I decided to take it out.

To calculate the IDI option price, I used the Black Model with the assumption that the IDI is $IDI_t = 100,000$ in each day of the sample.

3 Model

Fix a probability space $(\Omega, \mathcal{F}_t, P)$ and a filtration $(\mathcal{F}_t)$, satisfying the usual conditions, and suppose that $X(t)$ is a Markov process in some state space $D \subset \mathbb{R}^n$, which evolves according to the following SDE

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t) + \int_{\mathbb{R}^n} Y M(dy, dt) \tag{3}$$

where W is an $(\mathcal{F}_t)$-standard Brownian Motion in $\mathbb{R}^n$; $\mu : D \to \mathbb{R}^n$, $\sigma : D \to \mathbb{R}^{n\times n}$, and M is a Jump Measure, whose intensity is given by μ. We suppose that this intensity measure can be decomposed in the following manner $\mu(dy, dt) = \nu(dy)dt = \lambda(X(t))f(dy)dt$ [2]. $f(y)$ is the jump fixed probability distribution on $\mathbb{R}^n$ and, $\lambda(X(t))$ is the jump-intensity, for some $\lambda : D \to [0, \infty)$.

We fix an affine discount-rate function $R : D \to \mathbb{R}$ and impose an affine structure on μ, $\sigma\sigma^T$, and λ:

- $\mu(x) = K_0 + K_1 x$, for $K = (K_0, K_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}$.
- $(\sigma(x)\sigma(x)^T)_{ij} = (H_0)_{ij} + (H_1)_{ij} x$, for $H = (H_0, H_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n}$.
- $\lambda(x) = l_0 + l_1 x$, for $l = (l_0, l_1) \in \mathbb{R} \times \mathbb{R}^n$.
- $R(x) = \rho_0 + \rho_1 x$, for $\rho = (\rho_0, \rho_1) \in \mathbb{R} \times \mathbb{R}^n$.

For $c \in \mathbb{C}^n$, define $\theta(c) = \int_{\mathbb{R}^n} \mathrm{e}^{cy} f(dy)$. This jump transform θ determines the jump-size distribution. The coefficients (K, H, l, θ) of X completely determine its distribution, given an initial condition $X(0)$.

[2] With this hypothesis the Poisson Random Measure is equivalent to a Compound Poisson Process with intensity $\lambda(X(t))$ and jump-size fixed probability distribution $f(y)$.

3.1 Princing Bonds

The price of a default-free bonus with time to maturity h is given by:

$$P(h, X(t)) = \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-\int_t^{t+h} R(X(s))ds} \mid \mathcal{F}_t\right) \tag{4}$$

Note that the expectation is taken under the risk-neutral measure $\mathbb{Q}$, where all discounted prices are martingals.

To relate neutral-risk to physical measures, suppose that under the risk-neutral measure $\mathbb{Q}$ the parameters of (3) are given by $\Theta^{\mathbb{Q}} = (K^{\mathbb{Q}}, H^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, \theta^{\mathbb{Q}})$, under the physical measure $\mathbb{P}$, $\Theta^{\mathbb{P}} = (K^{\mathbb{P}}, H^{\mathbb{P}}, \lambda^{\mathbb{P}}, \theta^{\mathbb{P}})$, and under the brownian motion physical measure $\mathbb{P}^{MB}$, $\Theta^{\mathbb{P}^{MB}} = (K^{\mathbb{P}}, H^{\mathbb{P}}, \lambda^{\mathbb{Q}}, \theta^{\mathbb{Q}})$[3]. Then, the relation between these measures are given by the following Radon-Nikodym derivative.

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \frac{d\mathbb{P}}{d\mathbb{P}^{MB}} \times \frac{d\mathbb{P}^{MB}}{d\mathbb{Q}} \tag{5}$$

The first term defines the change of measures of Poisson Random Measures, whereas the second the change of measures of Brownian Motions.

$$\frac{d\mathbb{P}}{d\mathbb{P}^{MB}} = \exp\left(\int_{[0,T]\times\mathbb{R}^n} (\lambda^{\mathbb{Q}}(s)f^{\mathbb{Q}}(dy, ds) - \lambda^{\mathbb{P}}(s)f^{\mathbb{P}}(dy, ds)) + \sum_{i=1}^{N_T} \phi(Y_i, \tau_i)\right) \tag{6}$$

where $\phi(Y_i, \tau_i) = \ln \frac{\lambda^{\mathbb{P}}(\tau_i)df^{\mathbb{P}}(Y_i,\tau_i)}{\lambda^{\mathbb{Q}}(\tau_i)df^{\mathbb{Q}}(Y_i,\tau_i)}$ and τ_i is the i-th jump-time. Note that with this specification the intensity can be stochastic and jump-size distribution can be time varying [4].

$$\frac{d\mathbb{P}^{MB}}{d\mathbb{Q}} = \mathcal{E}\left(-\int_0^T \gamma(s)ds\right) \tag{7}$$

where $\mathcal{E}$ is the stochastic exponential defined as $\mathcal{E}(X(t)) = \exp(X(t) - [X, X](t))$, $[X, X]$ is the total quadratic-variation process, and $\gamma(t)$ is the market price of risk of the form Extended Affine (see Cheridito et al. (2007)).

[3] Note that ρ is not modified by the change of measures.

[4] If the Poisson Random Measure is a Compound Poisson Process with constant intensity and invariant jump-size distribution, then the Radon-Nikodym derivative is

$$\frac{d\mathbb{P}}{d\mathbb{P}^{MB}} = \exp\left(T(\lambda^{\mathbb{Q}} - \lambda^{\mathbb{P}}) + \sum^{N_T} \ln \frac{d\nu^{\mathbb{P}}(Y)}{d\nu^{\mathbb{Q}}(Y)}\right)$$

$$\gamma(t) = (\sigma(X(t))\sigma(X(t))^T)^{-1}((K_0^{\mathbb{P}} - K_0^{\mathbb{Q}}) + (K_1^{\mathbb{P}} - K_1^{\mathbb{Q}})X(t)).$$

Under technical conditions the price of a bonus is given below,

$$P(h,x) = \mathrm{e}^{A(h)+B(h)^T x} \tag{8}$$

where A and B satisfy the Ricatti's ODE

$$\begin{aligned} \dot{B}(h) &= \rho_1 - K_1^T B(h) - \frac{1}{2}B(h)^T H_1 B(h) - l_1(\theta(B(h)) - 1), & (9)\\ \dot{A}(h) &= \rho_0 - K_0^T B(h) - \frac{1}{2}B(h)^T H_0 B(h) - l_0(\theta(B(h)) - 1), & (10) \end{aligned}$$

with boundary conditions $B(0) = 0$ and $A(0) = 0$. The ODE (9) – (10) is easily obtained by an application of Ito's Lemma to the candidate solution (8) of the bonus price (4) with the additional assumption that the expected instantaneous return of an asset is equal to the instantaneous risk free rate $R(x)$.

Bond yields can be found applying the following,

$$y(h,t) = \bar{A}(h) + \bar{B}(h) \cdot X(t) \tag{11}$$

where $\bar{A}(h) = -\frac{A(h)}{h}$ and $\bar{B}(h) = -\frac{B(h)}{h}$

3.2 Pricing Asian Options: Fourier Transform in the cdf

Here I show how to price Asian options that depend on the path of the short-term interest rate. Here, the path dependence will be denoted by the integral in time of the instantaneous interest rate.

$$Y(t,T) = \int_t^T r(u)du \tag{12}$$

The DI Index can be approximated by the following equation

$$IDI(T) = IDI(t)\mathrm{e}^{Y(t,T)} \tag{13}$$

It's possible to show that this approximation is a very good one.

The IDI call option has the following payoff structure:

$$\begin{aligned} C(K,t) &= \mathbb{E}\left\{\mathrm{e}^{-Y(t,T)}[IDI(T) - K]^+ \mid \mathcal{F}_t\right\} & (14)\\ C(k,t) &= \mathbb{E}\left\{[IDI(t) - \mathrm{e}^k \mathrm{e}^{-Y(t,T)}]^+ \mid \mathcal{F}_t\right\} & (15) \end{aligned}$$

Where $k = log(K)$ and for now on I will use $y = Y(t,T)$.

To price the IDI option, define the Fourier Transform $\mathcal{G}(\lambda, k)$ as:

$$\mathcal{G}(\lambda, k) = E[\mathrm{e}^{-y\lambda} I_{y \geq k - idi}(k) \mid \mathcal{F}_t] \tag{16}$$

Then, the option price is given by:

$$C(k,t) = IDI(t)\mathcal{G}(0,k) - \mathrm{e}^k \mathcal{G}(1,k) \tag{17}$$

Note that $\mathcal{G}(0,k) = P^{\mathbb{Q}}(Y \geq k - idi \mid \mathcal{F}_t)$ and $\mathcal{G}(\lambda,k) = cP^{\mathbb{F}}(Y \geq k - idi \mid \mathcal{F}_t)$ where the Radon-Nykodim derivative is given by $\frac{d\mathbb{F}}{d\mathbb{Q}} = \frac{\mathrm{e}^{-\lambda y}}{E(\mathrm{e}^{-\lambda y} \mid \mathcal{F}_t)}$ and $c = E(\mathrm{e}^{-\lambda y} \mid \mathcal{F}_t)$. Using this change of measure the option price can be rewritten as:

$$C(k,t) = IDI(t)P^{\mathbb{Q}}(Y \geq k - idi \mid \mathcal{F}_t) - \mathrm{e}^k P(t,T) P^{\mathbb{F}}(Y \geq k - idi \mid \mathcal{F}_t) \tag{18}$$

For a fixed λ, the generalized Fourier Transform of $\mathcal{G}$ is:

$$\hat{\mathcal{G}}(\lambda, u) = \int_{-\infty}^{+\infty} \mathrm{e}^{iuk} d\mathcal{G}(\lambda, k) \tag{19}$$

Solving the Levy Integral, we have:

$$\hat{\mathcal{G}}(\lambda, u) = -E[\mathrm{e}^{-y(\lambda - iu) + iu \cdot idi} \mid \mathcal{F}_t] \tag{20}$$

Using the Radon-Nykodim derivative we have that,

$$\hat{\mathcal{G}}(\lambda, u) = -E^{\mathbb{Q}}[\mathrm{e}^{-y\lambda} \mid \mathcal{F}_t] \times E^{\mathbb{F}}[\mathrm{e}^{iuy + iu \cdot idi} \mid \mathcal{F}_t] \tag{21}$$

Defining,

$$I = \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-iux} \hat{\mathcal{G}}(\lambda, u)}{iu} du \tag{22}$$

It's possible to show that

$$\mathcal{G}(\lambda, x) = \frac{\mathcal{G}(\lambda, -\infty)}{2} - I \frac{1}{2\pi} \tag{23}$$

where $\mathcal{G}(1, -\infty) = P(t,T)$ and $\mathcal{G}(0, -\infty) = 1$.

According to Joslin (2007), under the measure $\mathbb{F}$, y is roughly normally distributed. More precisely, $\hat{\mathcal{G}}$ can be rewritten as

$$\hat{\mathcal{G}}(\lambda, u) \approx c \times \mathrm{e}^{iu\mu_y - \frac{1}{2}u^2\sigma_y^2} \tag{24}$$

The Gaussian behavior under $\mathbb{F}$ happens because, despite the ratio of the Levy integrand is not constant (i.e. they are not the same), there is no huge variation, particularly in the area that is relevant for computing the integral numerically.

The implication of this result is that I only need to compute the first and second moments to calculate the IDI option price.

This procedure, contrary to the Fast Fourier Transform method of Carr and Madan (1999), is quite fast and allow me to price options of different moneyness in the whole time-series. Collin-Dufresne and Goldstein (2002) also present an approximated method based in cumulants to price swaptions in a fast way, but according to Joslin (2007) is less accurate and slower than the procedure presented here. Other works that price IDI options using different methods are Almeida and Vicente (2009a), Almeida and Vicente (2009b), and Almeida and Vicente (2008).

3.2.1 Change of Measure

Fix $\lambda = 1$, then the Radon-Nykodim derivative is given by:

$$M(t) = \frac{d\mathbb{F}}{d\mathbb{Q}} = \frac{\mathrm{e}^{-y}}{E(\mathrm{e}^{-y} \mid \mathcal{F}_t)} = \frac{\mathrm{e}^{-y}}{P(t,T)} = \mathrm{e}^{-y}\mathrm{e}^{-A(\tau)-B(\tau)X(t)} \tag{25}$$

where τ is the time to maturity of the option, $T = t + \tau$, and $A(\tau)$ and $B(\tau)$ are the solutions of the Ricatti's ODE.

By the Martingal Representation Theorem, we can find the market price of risk θ that relates $\mathbb{F}$ to $\mathbb{Q}$ by using the differential operator in $M(t)$:

$$dM(t) = -M(t)\theta(t)^T dW^{\mathbb{Q}}(t) \tag{26}$$

where,

$$\theta(t) = \Sigma(X(t),t)^T B(\tau) \tag{27}$$

Girsanov Theorem gives the relation between brownian motions under different measures:

$$dW^{\mathbb{F}}(t) = \theta(t)dt + dW^{\mathbb{Q}}(t) \tag{28}$$

Combining the results of these two theorems we can show that the state vector under the measure $\mathbb{F}$ follows:

$$dX(t) = (K_0^{\mathbb{F}} + K_1^{\mathbb{F}} X(t))dt + \Sigma(X(t),t)dW^{\mathbb{F}}(t) \tag{29}$$

with

$$
\begin{aligned}
K_0^{\mathbb{F}} &= K_0^{\mathbb{Q}} - \alpha \odot B(\tau) && (30)\\
K_1^{\mathbb{F}} &= K_1^{\mathbb{Q}} - \beta \odot B(\tau)\iota^T && (31)\\
\Sigma(X(t),t) &= \sqrt{diag(\alpha + \beta X(t))} && (32)
\end{aligned}
$$

where $\odot$ is defined as the element by element product.

4 Empirical Analysis

4.1 Estimation Procedure

In order to estimate the models I solve for $X(t)$ using the linear bond yields relation 11 for the maturities 1, 126 and 378 working days to find the factors.

$$
X(t) = \bar{B}^{*-1}[y^*(t) - \bar{A}^*] \tag{33}
$$

where $\bar{B}^* = [\bar{B}(1), \bar{B}(126), \bar{B}(378)]^T$, $\bar{A}^* = [\bar{A}(1), \bar{A}(126), \bar{A}(378)]^T$, and $y^*(t) = [y(1,t), y(126,t), y(378,t)]^T$.

The vector of yields $y^*(t)$ used in the inversion are assumed to be priced without error, while the other yields $y^{**}(t)$ and options $C^{**}(t)$ are assumed to be priced with Gaussian independent errors. Note that I use the notation $(*)$ for yields that are exactly priced, $(**)$ for yields and options that are price with error, and no star for actual yields and option prices $(y(t), C(t))$.

The likelihood is built with all the yields and options (for the models that are estimated with them). To construct the likelihood for the models that account for jumps I assume that only one jump can happen by the end of each day. In other words, I approximate a Poisson process by a Binomial process. With this hypothesis the time-t likelihood is given by a mixture of two normal distributions, with weights given by the jump intensity $\lambda(X(t))$:

$$
\begin{aligned}
l_t(\Theta) &= (1 - \lambda(X(t-1))h) f(y(t); \Theta \mid N(t) - N(t-1) = 0)\\
&+ \lambda(X(t-1))h f(y(t); \Theta \mid N(t) - N(t-1) = 1) \qquad (34)
\end{aligned}
$$

where $h = \frac{1}{252}$ [5] and $f_t = f(y(t) \mid N(t) - N(t-1))$

[5] In Brazil we considered that one year has 252 working days, and the counting rule is $\frac{wd}{252}$.

$$
\begin{aligned}
f_t &= f(y^*(t) \mid N(t)-N(t-1)) \times f(y^{**}(t) \mid N(t)-N(t-1)) \\
&= |\bar{B}^*|^{-1} f(X(t) \mid X(t-1), N(t)-N(t-1)) \times f(y^{**}(t))
\end{aligned} \quad (35)
$$

is a multivariate normal probability density function (pdf).

I use a MCMC Metropolis Hasting algorithm to estimate all the models. I draw $100,000$ set of parameters from the posterior distribution an use the mean of the last $5,000$ as the estimates of the models. Details of the specification of the model are given in appendix A.

4.2 Model implied options and Factors

4.2.1 Options

I do a race horse between the models that are estimated without using derivatives, to evaluate which of them best price the IDI call options. Figure 3 plots the evolution of the ITM, ATM and OTM IDI call prices of maturity 240 wd. The elected models are the A30 and the A30J. Although the A30 is the best in terms of average mean absolute pricing errors, the A30J is better in capturing extreme spike behaviors in prices. The A30JT overprice the options of all moneyness.

About the models that incorporate options in the estimation (A30o, A30oJ and A30JT), it is possible to say that there is a conflict between fitting ATM and OTM options. The ATM derivatives are much better priced than the OTM ones. Interestingly, none of the three models are able to capture the huge spike in prices around $Jun-06$, especially for the OTM options.

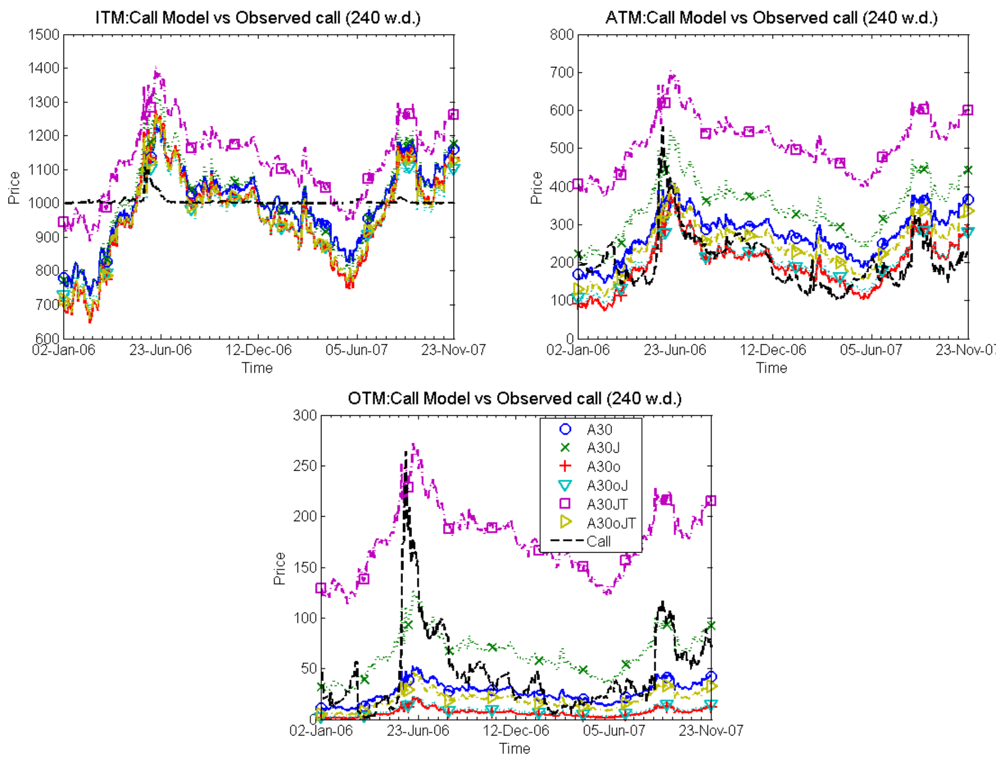

Figure 3: The daily evolution of the implied call IDI option prices with maturity 240 working days and moneyness of $(0.99, 1, 1.01)$.

4.2.2 Factors

Interpreting the factors just looking at them in picture 4 are quite hard. In principle, they are the solution implied by the system 33.

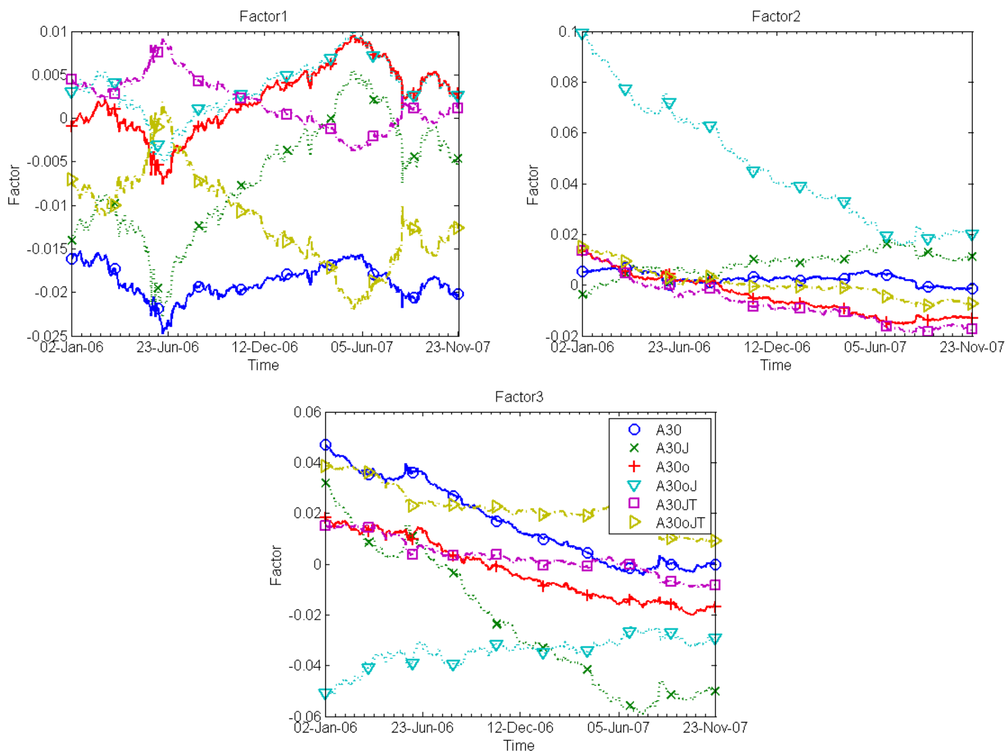

Figure 4: The daily evolution of the three factors $X(t) = \bar{B}^{*-1}[y^*(t) - \bar{A}^*]$ inverted from yields of maturities 1, 126 and 378.

In the search for an interpretation for the factors I construct the empirical level ($L(t)$), slope ($S(t)$) and curvature ($C(t)$) factors as it follows

$$L(t) = \frac{y(63,t) + y(252,t) + y(504,t)}{3} \tag{36}$$

$$S(t) = y(504,t) - y(63,t) \tag{37}$$

$$C(t) = y(504,t) + y(63,t) - 2 \cdot y(252,t) \tag{38}$$

and run regressions with the model implied factors as explained variables and the empirical factors as the explanatory variables:

$$X_i(t) = a + b_1 L(t) + b_2 S(t) + b_3 C(t) + \varepsilon_i(t) \tag{39}$$

The results of the regressions are plotted in figure 5. Note that in all models (except for the A30) there is factor that is the average of the empirical ones. In the models without jumps (top two graphs), factor 3 is explained by the level and curvature, but the first is more important in the one estimated without options. This bigger curvature loading happens in all implied factor 3 from the models that use options in the estimation. One possible explanation for it is that curvature is closely related to variance, and the use of options allows to better identify volatility. Contrary to the models that do not present jumps, the factors of the models with jumps are almost equal to the ones with options and jumps, except for a rotation in the constant intensity models (see the last four graphs). This finding indicates that models with jumps have the factors better identified.

Factor 3 is the only one that is allowed to jump. In order to evaluate the probability of jumps, I decompose this factor in two parts: (1) continuous part $X_3^c(t)$, and (2) a jump part $N(t)$. So, $X_3(t)$ can be written as:

$$X_3(t) = X_3^c(t) + N(t) \tag{40}$$

After that, I compute the probability of one jump-event in the third state between periods t and $t-1$ given all available yields information $y(t)$ up to time t, $P(N(t) - N(t-1) = 1 \mid Y(t))$. Figure 6 presents the probability of jumps for the models with constant intensity (line), along with changes in the Selic monetary policy interest rate target (circles). Note that, although there are more jumps happening in the model with options, changes in the target are more closely tracked by the model with the model without options. One explanation is that jumps are linked to several geopolitical events and surprises in macroeconomic releases, not related with monetary policy decisions, that generally increases the uncertainty over the economy. As options capture market volatility better than bonds, jumps not related with policy decisions are better identified in models estimated with option data.

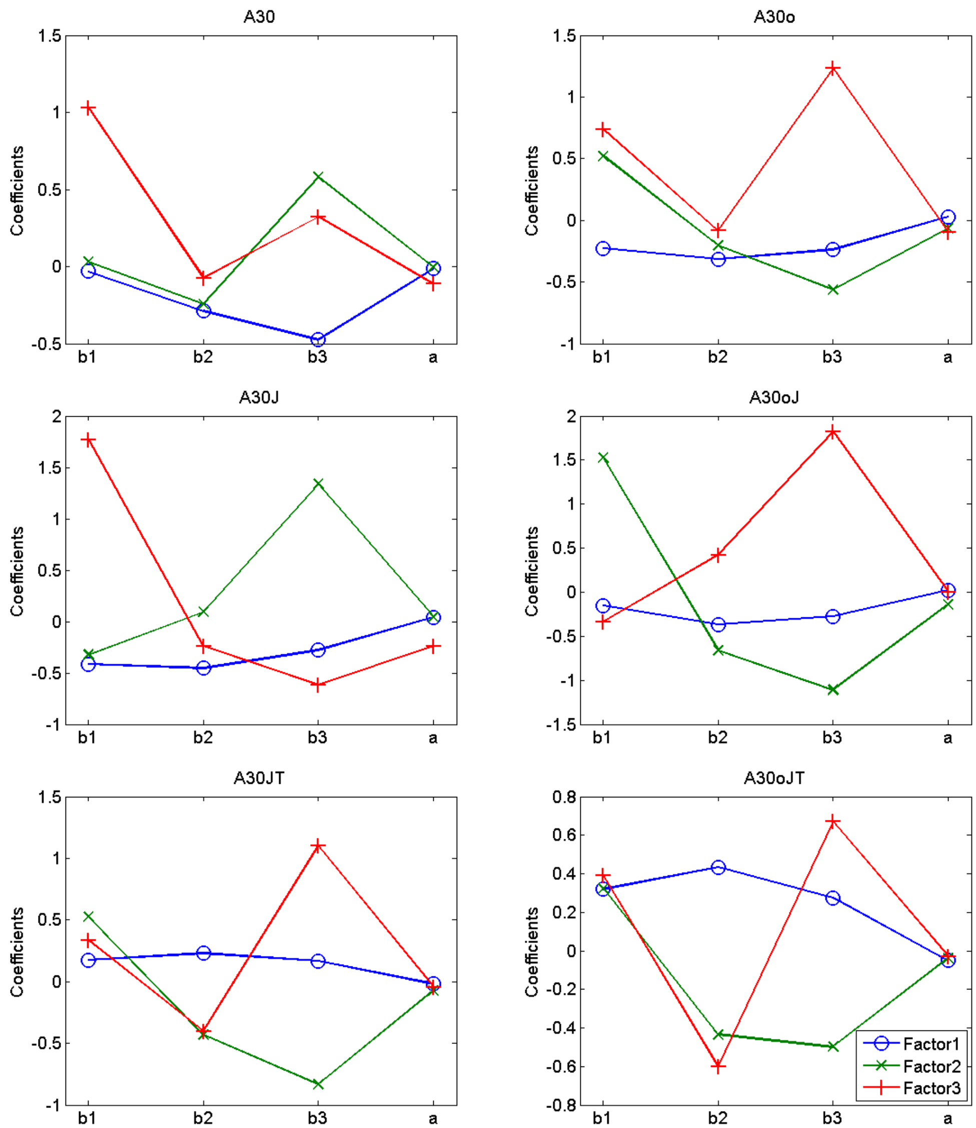

Figure 5: The coefficients of the factor regressions done as in 39 for the six models.

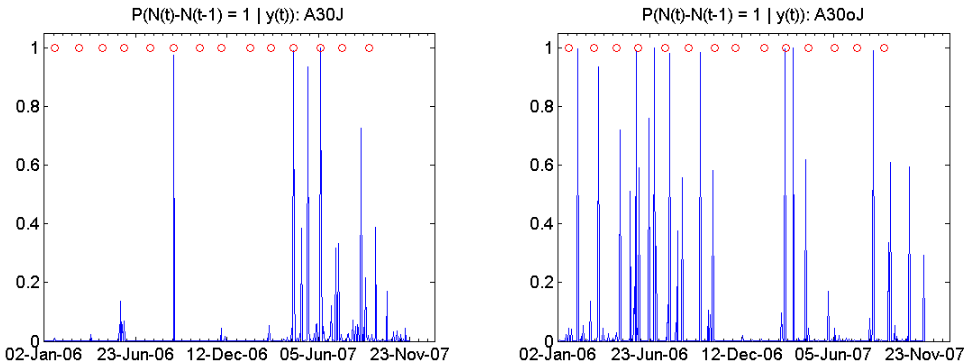

Figure 6: Circles are changes in the Selic monetary policy interest rate target. The line is the probability of one jump-event in the third state between periods t and $t-1$ given all available yields information $y(t)$ up to time t, $P(N(t) - N(t-1) = 1 \mid Y(t))$.

5 Risk Premium Decomposition

As Joslin (2007) shows, the T-year zero coupon yield $y(t,T)$ can be decomposed into tree components: (1) physical expectation of future interest rates, (2) convexity and, (3) risk premia. Within the inclusion of jumps we can develop the latter component and split it into two: jump risk premia and brownian risk premia.

$$y(t,T) = y^E(t,T) + y^{BRP}(t,T) + y^{JRP}(t,T) + y^C(t,T) \tag{41}$$

$$\begin{aligned}
y^E(t,T) &= \frac{1}{T}\int_t^{t+T} \mathbb{E}^P[r(u)]du && (42)\\
y^{BRP}(t,T) &= \frac{1}{T}\int_t^{t+T} (\mathbb{E}^{MB}[r(u)] - \mathbb{E}^P[r(u)])du && (43)\\
y^{JRP}(t,T) &= \frac{1}{T}\int_t^{t+T} (\mathbb{E}^{Q}[r(u)] - \mathbb{E}^{MB}[r(u)])du && (44)\\
y^C(t,T) &= \frac{1}{T}\left(\ln \mathbb{E}^Q[\mathrm{e}^{\int_t^{t+T} r(u)du}] - \int_t^{t+T} \mathbb{E}^Q[r(u)]du\right) && (45)
\end{aligned}$$

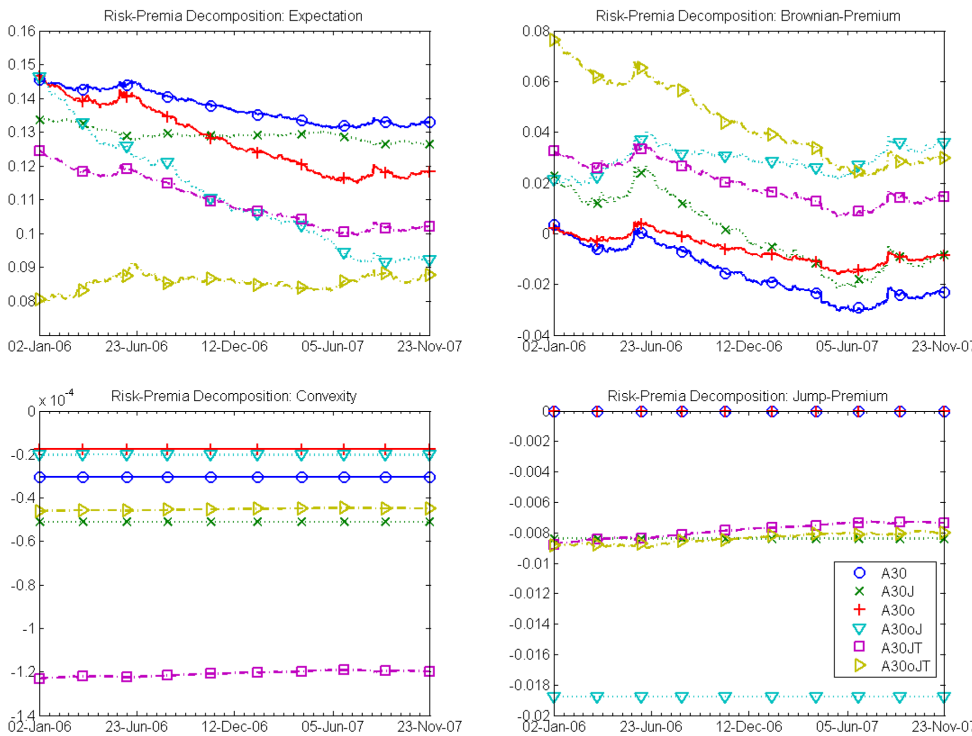

Figure 7: The daily evolution of the one year risk premiums, expectation and Convexity measures given by equations 42-45.

The Theory of Expectations postulate that agents are risk neutral. This assumption implies that long term rates are equal to the average future expected short-term interest rates. Mathematically speaking, the expectation can be calculated using different probability measures, yielding distinct values.

In one hand, except for a small and insignificant convexity term, the theory of expectations holds under the risk neutral measure. On the other hand, the right measure to undertake the estimation is the physical one. This allows to naturally define risk premia as the difference of expectations taken under the risk neutral and the physical measure. The decomposition proposed in equations 42-45 uses this intuitive definition of risk premia.

Figure 7 plots all the terms of the decomposition. Note first, that the Brownian premium of the models estimated with options are greater in levels than theirs only yields counterparts. This difference seems to be more prominent at the end of the sample (except for the models with time-varying intensity). The opposite behavior is true for the expectation component. Second, models with options have more flat risk premiums, indicating a greater adherence to the weak form of the expectation hypothesis [6]

The jump risk-premia is allways negative, which, at first sight, is at odds with the intuition that jumps in interest rates are risky and so should have a positive premium. But, in my sample, jump sizes have a negative mean (μ_J), therefore the arrival of a jump is always a good news to bond investors because their returns are increased with the fall of interest rates.

Another finding is that the negative jump premia is compensated by greater levels of brownian premium, this can be seen when comparing models with jumps with theirs counterparts that do not jump (see graphs in the right of picture 7). Probably with a sample that includes negative and positive jumps would turn the sign of the jump premia.

Last, the convexity term is very small and influences the actual yield in less than 1.5 basis point, indicating that the risk neutral measure is consistent with the Expectation Hypothesis.

6 Conclusion

In this paper I estimate six Gaussian Affine Term Structure Models with jumps and options in order to assess two questions: (1) what are the implications of incorporating jumps in an ATSM for Asian option pricing, and (2) how jumps and options affect the bond risk-premia dynamics.

[6] Long yields are equal to future expected short rates plus a constant risk-premium (that can vary with bond maturity).

I find that: (1) jump risk-premia is negative in a scenario of decreasing interest rates and explain 10%-20% of the level of yields, (2) Asian options help to reconcile, in part, the weak form of Expectation Hypothesis, and (3) gaussian models without jumps and with constant intensity jumps are good to price the IDI Asian options.

I do not analyze the option pricing and risk-premia implications for models with stochastic volatility and/or models where some of the factors are observable. One interesting extension would be to evaluate these implications for the Piazzesi (2005) model.

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A Specification of the models

A.1 A30

$$K_1^P = \begin{bmatrix} K_{1,11}^P & 0 & 0 \\ K_{1,21}^P & K_{1,22}^P & 0 \\ K_{1,31}^P & K_{1,32}^P & K_{1,33}^P \end{bmatrix}, \quad K_1^Q = \begin{bmatrix} K_{1,11}^Q & 0 & 0 \\ K_{1,21}^Q & K_{1,22}^Q & 0 \\ K_{1,31}^Q & K_{1,32}^Q & K_{1,33}^Q \end{bmatrix} \tag{46}$$

$$K_0^P = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad K_0^Q = \begin{bmatrix} K_{0,1}^Q \\ K_{0,2}^Q \\ K_{0,3}^Q \end{bmatrix} \tag{47}$$

$\rho_0 \in \Re$, $\rho_1 = [1,1,1]$, $H_{0,ij} \in \Re_+$ for $i = j$ and 0 otherwise, $H_{1,ij} = [0,0,0]$ for $i,j = 1,2,3$, $l_0^Q = l_0^P = 0$, and $l_1^Q = l_1^P = [0,0,0]$.

A.2 A30J

Equal to A30, except for $l_0^i \in \Re_+$ for $i = Q, P.$, and $f^i(y)$ is gaussian with mean μ_J^i and variance Σ_J^i for $i = Q, P.$.

$$\Sigma_J^P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Sigma_{J,33}^P \end{bmatrix}, \quad \Sigma_J^Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Sigma_{J,33}^Q \end{bmatrix} \tag{48}$$

$$\mu_J^P = \begin{bmatrix} 0 \\ 0 \\ \mu_{J,3}^P \end{bmatrix}, \quad \mu_J^Q = \begin{bmatrix} 0 \\ 0 \\ \mu_{J,3}^Q \end{bmatrix} \tag{49}$$

$\mu_{J,3}^i \in \Re$ and variance $\Sigma_{J,33}^i \in \Re_+$ for $i = Q, P.$.

A.3 A30JT

Exactly the same as A30J, but with $l_1^i \in \Re_+^3$ for $i = Q, P.$.