An Intensity-based Model for Pricing Variable Coupon Bonds

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Abstract

This paper presents a reduced form model for the valuation of variable-coupon bonds where the coupon rate fluctuates with the credit rating of the issuing firm. We work within a class of intensity based pricing models where a Cox (or a doubly stochastic Poisson) process governs the intensity of the ratings change. The time-variation in the credit transition process is modeled via a continuous-time inhomogeneous Markov chain. With a desire to avoid making strong assumptions on the properties of the generator matrix, we develop a general recursive pricing model. As a special case, we derive essentially closed form solutions for the prices of step-up only bonds within an affine term structure setting.

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1 Introduction

Recently there has been increased activity in the issuance of variable coupon bonds where the coupon rate rises and falls with the credit rating of the issuing company. Several issuers, most notably the European telecom companies Deutsche Telekom, British Telecom, France Telecom, KPN and Telecom Italia have issued debt with a ratings sensitive coupon provision where the coupon rate rises or falls in response to changes in a company's credit rating. For example, KPN issued a step-up *only* bond where the coupon rate increases should the bond be downgraded to a trigger credit rating. AT&T issued a bond with a step-up provision that would pay a 25 basis point increase in the coupon rate whenever its credit rating is downgraded by a ratings agency. Other bond issuers also allow for a step-down provision in the coupon rate should the firm's credit rating improve. For example, in 2000, Deutsche Telekom issued bonds with ratings-sensitive coupons that would step-up by 50 basis points should Moodys and S&P both lower their credit rating. The bond also contained a step-down provision if their credit rating was raised by both agencies.

Examples of intensity-based reduced form models for pricing bonds subject to default risk are well established in the literature.[1] Jarrow, Lando, and Turnbull (1997) utilize a doubly stochastic Poison (or Cox) process with a time homogeneous transitions matrix to model prices of defaultable securities. Lando (1998), in reporting earlier studies by Fons and Kimball (1991), note that default rates of firms show significant time variation over time. Thus, a more realistic method of modeling ratings intensities would allow for this variation in the intensity process. Lando (1998) generalizes the model proposed in Jarrow, Lando, and Turnbull (1997) by allowing the transition intensities to vary with the underlying state vector and derives a method for pricing credit sensitive debt. Duffee (1998) documents a negative correlation between credit spreads and Treasury yields. Noting that a generator matrix with constant intensities cannot replicate this correlation, Lando (1998) allows both the short rate, $r(X_t)$, and the transition intensities, $\Lambda(X_t)$, to be functions of a vector of state variables X_t. However, the closed-form solutions obtained hinge on the assumption that the generator matrix has the representation $\Lambda_t = B\mu(X_t)B^{-1}$ where $\mu(X_t)$ is a diagonal matrix and B is a KxK matrix whose columns consist of the eigenvectors of Λ_t. The imposition of this structure on the generator matrix requires a strong assumption that the eignevectors exist and are not time varying.

This paper seeks a method for the pricing of credit-sensitive variable coupon bonds, without resorting to the aforementioned assumption on the generator matrix while still working within the generalized framework of Lando (1998). Hence, a version of a doubly stochastic Poisson process is utilized in modeling the transition times where the intensity process is assumed to be governed by an inhomogeneous continuous time Markov chain. The innovation involves recasting the more difficult credit-sensitive problem into a more familiar problem of pricing defaultable securities by mapping the time of the next ratings change with the default time and the price of the bond at the next transition time (conditioned on the new credit rating) with the recovery value in a standard reduced-from default setting. This crucial insight underpins the setting of our recursive pricing framework, which albeit elegant has fallen short of the ideal of obtaining a closed form solution. However, for the special case of step-up only bonds with 3 and 4 *coupon* classes, assumed knowledge of coupon class transition intensities plus the imposition of standard affine assumptions on the model lead to essentially closed form solutions for bond prices. Recent work on pricing bonds with ratings-based coupons include Lando and Mortensen (2005), Vorst, Houweling, and Mentink (2004) and Das, Acharya, and Sundaram (2002).[2] These models share a common discrete-time framework where the ratings transitions are characterized by a time-homogeneous Markov chain, hence they are not explicitly designed to capture time variation in intensities or the negative correlation between spreads and interest rates.

[1]See Duffie and Singleton (2003) for an overview.

[2]Manso, Strulovici, and Tchistyi (2005) study properties of step-up bonds in an endogenous default setting and find them to be inefficient instruments from the perspective of the debt issuer.

The paper is organized as follows. Section 2 carefully constructs a recursive theoretical pricing formula derived for the general case affording both step-up and step-down provisions. Special cases for step-up only bonds are considered in Section 3 where we derive essentially closed form solutions when relevant functions are taken to be affine in the underlying state variables. Section 5 concludes.

2 General Recursive Pricing Model

We begin our analysis by looking at the general case of a variable-coupon bond where the coupon payment can both step-up in response to a ratings downgrade and step-down in response to a ratings upgrade.

2.1 General specifications and theory

We fix a probability space $(\Omega, \mathcal{F}, P)$ and filtrations $\{\mathcal{F}_t\}$,$\{\mathcal{H}_t\}$ and $\{\mathcal{G}_t\}$ satisfying the *usual conditions* where $\mathcal{F}_t \subset \mathcal{H}_t \subset \mathcal{G}_t$. These filtrations are formally defined later in the paper.[3] The existence of an equivalent martingale measure (i.e., a risk-neutral probability measure) Q is assumed implying the absence of arbitrage under reasonable restrictions on trading strategies.[4] Unless otherwise specified, all expectations, intensities and probabilities are assumed to be those under this equivalent martingale measure Q. We take as given an underlying N-dimensional $\mathcal{F}_t$-adapted vector of risk factors $\{X_t\}$ and assume that the short-rate takes the form $r_t = r(X_t)$.[5] The intensity process is modeled via a continuous time, inhomogeneous Markov chain on the state space $S = \{1, 2, \ldots, K\}$. Each element of the state space S represents a different credit class with 1 denoting the class with the highest possible credit rating and K denoting default. The time-varying credit process is formally captured by $\{i_t\}$, a $\mathcal{G}_t$-adapted right-continuous process defined over the state space S. However, when clear from the context, we suppress the subscript and simply use i to refer to the "current state" and the random variable j to refer to the "new state" conditional on the next transition. We assume the existence of a generator matrix representing risk-neutral intensities. We borrow from Lando (1998) and adopt the following generator matrix given under the risk-neutral measure

$$\Lambda(t) = \begin{pmatrix} -\lambda_{1t} & \lambda_{12t} & \lambda_{13t} & \ldots & \lambda_{1Kt} \\ \lambda_{21t} & -\lambda_{2t} & \lambda_{23t} & \ldots & \lambda_{2Kt} \\ \lambda_{31t} & \lambda_{32t} & -\lambda_{3t} & & \lambda_{3Kt} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix},$$

where the off-diagonal elements represent the non-negative transition rates between credit classes. Representing default, the absorbing state K is imposed by the last row of zeros. For simplicity of notation we suppress the dependence of the intensity parameters, $\Lambda(t) \equiv \Lambda(X_t)$, on the vector of underlying risk factors, X_t, as in the generator matrix in Lando (1998), who generalized the generator matrix of Jarrow, Lando, and Turnbull (1997). When there is a one-to-one correspondence between credit classes and coupon rates, one may alternatively choose to think of this matrix as the transition rates between coupon classes. The usefulness of this interpretation will become evident in the next section.

At each time t, the rate at which the process transitions out of state i is denoted by

[3]The *usual conditions* are said to be satisfied for a filtration $\{\mathcal{G}_t\}$ if $\mathcal{G}_0$ contains all the null sets of $\mathcal{F}$ and for all t, $\mathcal{G}_t$ is right-continuous, i.e., $\mathcal{G}_t = \mathcal{G}_{t+}$ holds for every $t \geq 0$.

[4]See Harrison and Kreps (1979).

[5]The risk factors may be latent or observable, and include systemic factors such as macroeconomic variables or interest rate factors as well as idiosyncratic factors particular to a specific industry or firm.

$$\lambda_{it} = \sum_{j \neq i}^{K} \lambda_{ijt}, \qquad i = 1, \dots, K \tag{1}$$

We impose the technical condition that for all t

$$\int_0^t \lambda_{is} ds < \infty \quad a.s. \tag{2}$$

We denote $\tau_t = \inf\{s > t : i_s \neq i_t\}$ to be the first time after the current time t that the process exits the current credit class. For any two sigma fields $\mathcal{G}_t$ and $\mathcal{F}_s$, $\mathcal{G}_t \vee \mathcal{F}_s$ is defined to be the smallest sigma field containing all the sets in $\mathcal{F}_s$ and all the sets in $\mathcal{G}_t$. The informational structure is stated as follows:

$$\begin{aligned} \mathcal{F}_t &= \sigma(X_s : 0 \leq s \leq t) & (3)\\ \mathcal{H}_t &= \mathcal{F}_t \vee \sigma(1_{\tau_s = v} : 0 < s \leq v \leq t) & (4)\\ \mathcal{G}_t &= \mathcal{H}_t \vee \sigma(i_s : 0 \leq s \leq t) & (5) \end{aligned}$$

Informally, $\mathcal{G}_t$ is the filtration containing all relevant information up to and including time t. Conditional on $\mathcal{H}_t$, one knows the timing of the transitions but not the path of the credit class process: the credit rating at time s is not known given the information in $\mathcal{H}_s$. Thus credit class information up to and including time t is $\mathcal{G}_t$-measurable but not $\mathcal{H}_t$-measurable, while transition times are $\mathcal{H}_t$-measurable but not $\mathcal{F}_t$-measurable. By construction of our filtrations we know that the first exit time τ_t is an $\mathcal{H}_s$-stopping time but not an $\mathcal{F}_s$-stopping time: the event $\{\tau_t = s\}$ is not measurable with respect to $\mathcal{F}_s$.

We model τ_t as the first jump time of a doubly stochastic counting process $\{N_{i,t}\}$ driven by the filtration $\{\mathcal{F}_t\}$. We provide the following definition appropriate to our setting.[6]

Definition 1 *For each state $i \in S$, the counting process $\{N_{i,t}\}$ is a doubly stochastic Poisson process driven by the filtration $\{\mathcal{F}_t\}$ if the generator matrix $\Lambda(t)$ is $(\mathcal{F}_t)$-predictable, and if for all t and $s > t$, conditional on the sigma field $\mathcal{G}_t \vee \mathcal{F}_s$, $N_{i,s} - N_{i,t}$ has a Poisson distribution with the parameter given by $\int_t^s \lambda_{iu} du$.*

So conditional on a particular realization of the intensity process, $\{N_{i,t}\}$ is a nonstationary or inhomogeneous Poisson process with a time varying intensity parameter. Using this notation, we can formally state the first time to exit the current state i by

$$\tau_t = \inf\{s > t : N_{i,s} > N_{i,t}\}. \tag{6}$$

This first transition time is an exponential random variable exhibiting the corresponding memoryless property. It follows that the conditional probability of no ratings transitions from time t to time s is given by[7]

[6] See Appendix I of Duffie (2001) for more details on doubly stochastic counting processes. Note that in our definition there exists an underlying counting process for each credit state i.

[7] The conditional first jump time has an exponential distribution with CDF equal to $Q(\tau \leq s | \mathcal{G}_t \vee \mathcal{F}_s) = 1 - e^{-\int_t^s \lambda_{iu} du}$. By differentiating this CDF with respect to s, we obtain the density function $f(\tau_t = \tau | \mathcal{G}_t \vee \mathcal{F}_s) = e^{-\int_t^\tau \lambda_{iu} du} \lambda_{i\tau}$.

$$
\begin{aligned}
Q(\tau_t > s|\mathcal{G}_t \vee \mathcal{F}_s) &= Q(N_{i,s} - N_{i,t} = 0|\mathcal{G}_t \vee \mathcal{F}_s) && (7)\\
&= e^{-\int_t^s \lambda_{iu} du}. && (8)
\end{aligned}
$$

Under the risk-neutral measure Q the probability that the first transition time is greater than s is given by

$$
\begin{aligned}
Q(\tau_t > s|\mathcal{G}_t) &= E\left[1_{\tau_t > s}|\mathcal{G}_t\right] && (9)\\
&= E\left[E[1_{\tau_t > s}|\mathcal{G}_t \vee \mathcal{F}_s]|\mathcal{G}_t\right] && (10)\\
&= E[Q[\tau_t > s|\mathcal{G}_t \vee \mathcal{F}_s]|\mathcal{G}_t] && (11)\\
&= E\left[e^{-\int_t^s \lambda_{iu} du}|\mathcal{G}_t\right] && (12)
\end{aligned}
$$

We note that the coupon payment C_i, is a deterministic function of the credit rating of the firm and denote by $C \in \mathbf{R}^K$ the vector of coupon payments where the last element C_K is set equal to 0. Taking an i-class bond paying a random coupon of C_i at a time s, we recursively denote the value of this payment at time $t < s$ as a discounted expectation under the equivalent martingale measure

$$
P_{t,s}^i = E\left[e^{-\int_t^s r_u du} 1_{(\tau > s)} C_i|\mathcal{G}_t\right] + E\left[e^{-\int_t^\tau r_u du} 1_{(\tau < s)} P_{\tau,s}^j|\mathcal{G}_t\right]. \quad (13)
$$

The first term captures the case where with probability $Q(\tau > s|\mathcal{G}_t)$ there are no jumps in the credit rating process and the bond holder receives a payment of C_i at time s. With probability $Q(\tau < s|\mathcal{G}_t)$, there is a jump in the credit rating process at which time the new value of the bond is given by $P_{\tau,s}^j$. One can think of this new price $P_{\tau,s}^j$ as a random "recovery value" to be gained at the first transition time of the credit rating process. Conditional on τ and the new state j, $P_{\tau,s}^j$ is again a conditional expectation under the risk-neutral measure. In the case of default ($i = K$) we assume for simplicity that there is no recovery value, thus C_k=0 and K absorbing implies $P_{t,s}^K = 0$ for all t and s.

We proceed to obtain an expression for the price of a credit-sensitive coupon payment. Many of the techniques in the following derivations are based on results in Lando (1998).[8] We begin with the first term in (13)

$$
\begin{aligned}
E\left[e^{-\int_t^s r_u du} 1_{(\tau > s)} C_i|\mathcal{G}_t\right] &= C_i E\left[E\left[e^{-\int_t^s r_u du} 1_{(\tau > s)}|\mathcal{G}_t \vee \mathcal{F}_s\right]|\mathcal{G}_t\right] && (14)\\
&= C_i E\left[e^{-\int_t^s r_u du} E\left[1_{(\tau > s)}|\mathcal{G}_t \vee \mathcal{F}_s\right]|\mathcal{G}_t\right] && (15)\\
&= C_i E\left[e^{-\int_t^s r_u du} Q\left[(\tau > s)|\mathcal{G}_t \vee \mathcal{F}_s\right]|\mathcal{G}_t\right] && (16)\\
&= C_i E\left[e^{-\int_t^s r_u du} e^{-\int_t^s \lambda_{iu} du}|\mathcal{G}_t\right] && (17)\\
&= C_i E\left[e^{-\int_t^s (r_u + \lambda_{iu}) du}|\mathcal{G}_t\right]. && (18)
\end{aligned}
$$

In addressing the second term in (13) we first note that the conditional density for τ_t can be obtained by differentiating the CDF of the first jump time of an inhomogeneous Poisson process

[8]For a textbook treatment see Theorem 11J and Section 11K of Duffie (2001). In all that follows we assume that necessary conditions for interchanging the integral and expectations operators (by Fubini's Lemma) are satisfied. The calculations repeatedly rely on the law of iterated expectations.

$$f(\tau_t = \tau | \mathcal{G}_t \vee \mathcal{F}_s) = e^{-\int_t^\tau \lambda_{iu} du} \lambda_{i\tau}. \tag{19}$$

Conditional on initial state i and information in $\mathcal{H}_s$, the probability that the process transitions to j at time s is given by the following ratio[9]

$$Q(i_s = j | \mathcal{G}_t \vee \mathcal{H}_s) \quad = \quad 1_{(\tau_t = s)} \frac{\lambda_{ijs}}{\lambda_{is}} \qquad , i \neq j \tag{20}$$

hence by definition of τ being the first transition time, we have

$$Q(i_\tau = j | \mathcal{G}_t \vee \mathcal{H}_\tau) \quad = \quad \frac{\lambda_{ij\tau}}{\lambda_{i\tau}} \qquad , i \neq j. \tag{21}$$

This follows from the memoryless property of the exponential distribution (the only relevant information in determining the transition probability is the intensity matrix at the time of the transition) and the property that the conditional expectation of migrating to state j given a transition event is the ratio of the intensities given above.

The conditional joint distribution of a continuous random variable, τ_t, and a discrete random variable, i_τ, can be captured by the following bivariate function

$$\begin{aligned}
\phi(\tau, j | \mathcal{G}_t \vee \mathcal{F}_s) &\equiv Q(i_\tau = j | \mathcal{G}_t \vee \mathcal{H}_\tau \vee \mathcal{F}_s) f(\tau_t = \tau | \mathcal{G}_t \vee \mathcal{F}_s) && (22)\\
&= Q(i_\tau = j | \mathcal{G}_t \vee \mathcal{H}_\tau) f(\tau_t = \tau | \mathcal{G}_t \vee \mathcal{F}_s) && (23)\\
&= \frac{\lambda_{ij\tau}}{\lambda_{i\tau}} e^{-\int_t^\tau \lambda_{iu} du} \lambda_{i\tau} && (24)\\
&= \lambda_{ij\tau} e^{-\int_t^\tau \lambda_{iu} du} && (25)
\end{aligned}$$

Using this we obtain the following

$$\begin{aligned}
E\left[e^{-\int_t^\tau r_u du} 1_{(\tau < s)} P^j_{\tau,s} | \mathcal{G}_t \vee \mathcal{F}_s \right] &= \int_t^s \sum_{j \neq i}^K e^{-\int_t^w r_u du} P^j_{w,s} \phi(w, j | \mathcal{G}_t \vee \mathcal{F}_s) dw && (26)\\
&= \sum_{j \neq i}^K \int_t^s e^{-\int_t^w r_u du} P^j_{w,s} \lambda_{ijw} e^{-\int_t^w \lambda_{iu} du} dw && (27)
\end{aligned}$$

Thus, the second term in (13) can be rewritten as

$$\begin{aligned}
E\left[e^{-\int_t^\tau r_u du} 1_{(\tau < s)} P^j_{\tau,s} | \mathcal{G}_t \right] &= E\left[E\left[e^{-\int_t^\tau r_u du} 1_{(\tau < s)} P^j_{\tau,s} | \mathcal{G}_t \vee \mathcal{F}_s \right] | \mathcal{G}_t \right] && (28)\\
&= E\left[\sum_{j \neq i}^K \int_t^s e^{-\int_t^w r_u du} \lambda_{ijw} P^j_{w,s} e^{-\int_t^w \lambda_{iu} du} dw | \mathcal{G}_t \right] && (29)\\
&= \sum_{j \neq i}^K \int_t^s E\left[e^{-\int_t^w (r_u + \lambda_{iu}) du} \lambda_{ijw} P^j_{w,s} | \mathcal{G}_t \right] dw && (30)
\end{aligned}$$

[9] Recall that $\mathcal{G}_t$ tells us the current state i and $\mathcal{H}_s$ tells us if the first transition time is equal to s. Intuitively, if $(\tau_t \neq s)$ then $Q(i_s = j | \mathcal{G}_t \vee \mathcal{H}_s) = 0$ for $i \neq j$.

The general recursive pricing result for variable-coupon bonds with both a step-up and a step-down provision is summarized as follows.

Proposition 1 *Consider a credit-sensitive coupon vector $C = (C_1, C_2, \ldots, 0)^T$ to be paid at time s and let C_i denote the ith element of C. Suppose the initial rating of the firm is i at time t and the credit rating process is characterized by the generator matrix $\Lambda(t)$. Under technical conditions and assuming zero recovery upon default, the value of the variable coupon payment is given by*

$$P^i_{t,s} = C_i E\left[e^{-\int_t^s (r_u+\lambda_{iu})du} | \mathcal{G}_t\right] + \sum_{j\neq i}^{K} \int_t^s E\left[e^{-\int_t^w (r_u+\lambda_{iu})du} \lambda_{ijw} P^j_{w,s} | \mathcal{G}_t\right] dw. \tag{31}$$

The value of receiving the principal amount (normalized to 1) at maturity is given as a special case by defining $C = (1, 1, 1, \ldots, 1, 0)^T$. Any recovery value assumption on the face value of bond can be easily incorporated by setting $P^K_{t,s}$ equal to the fractional recovery value. Alternatively, by following Duffie and Singleton (1999) modifications could be made to implement a fractional recovery of market value (RMV) assumption. Note that a pure vanilla corporate bond (with no step-up language) is treated as a simple special case with $C = (c, c, c, \ldots, c, 0)^T$. The total value of a bond paying discrete, credit-sensitive coupons follows from summing the value of the coupon payments plus the value of receiving the principal at maturity.

2.2 Simplification of the generator matrix

We could simplify the generator matrix by assuming that bond ratings transitions follow a birth and death process where only a downgrade or an upgrade of a single credit rating is allowed at anytime. From a modeling standpoint this restriction is not unrealistic in that jumps in credit quality of two or more levels are quite rare under Q.[10] Nevertheless, the occurrence of multiple downgrades within a single reporting period is still consistent within our restricted model. We could simply explain a reported jump of several ratings classes by sequential transitions occurring within a single credit-reporting period where the agency only observes and reports the initial and final ratings. The impact of this simplification on the pricing error of the bond is expected to be minimal.

The generator matrix for this simplified process is given by

$$\Lambda(t) = \begin{pmatrix} -\lambda_{1t} & \lambda_{12t} & 0 & 0 & \ldots & 0 \\ \lambda_{21t} & -\lambda_{2t} & \lambda_{23t} & 0 & \ldots & 0 \\ 0 & \lambda_{32t} & -\lambda_{3t} & \lambda_{34t} & & 0 \\ \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix},$$

The general pricing formula given by Proposition 1 in the previous section continues to apply in this setting.

2.3 Discussion

The solution to asset pricing problems where transitions are governed by an inhomogeneous continuous-time Markov chain is complicated by the difficulty of computing (risk-neutral) transition probabilities.

[10]See for example the estimated generator matrix given in Table 4 of Jarrow, Lando, and Turnbull (1997).

Given a technology for computing the probability of being in state j at time s given initial state i, asset pricing problems of this type would be as simple as the time-homogenous case studied in Jarrow, Lando, and Turnbull (1997). However, absent very strong assumptions on the generator matrix, the technology for solving for these transition probabilities is presently unknown. Lando (1998) makes the assumption that the generator matrix has the representation $\Lambda_t = B\mu(X_t)B^{-1}$ where $\mu(X_t)$ is a diagonal matrix and the columns of the time invariant matrix B contain the eigenvectors of Λ_t. Realistic conditions for a time-varying matrix possessing time-invariant eigenvectors are difficult to imagine and are undoubtedly restrictive. Our approach proposes an alternative solution method that relies on a recursive pricing scheme absent any restrictions on the nature of the generator matrix. Although elegant in form, our proposition too has fallen short of the ideal, as even under simplifying assumptions on the stochastic nature of X_t a closed-form solution is not obtainable. To further exacerbate the issue, our proposition is not operational in its current form. However, all is not lost from an applications point of view. The recursive nature of the pricing relation lends itself to a discrete-time approximation which can then be solved by backwards recursion. Das, Acharya, and Sundaram (2002) derive a recursive representation for pricing credit-sensitive bonds in a discrete-time setting, however the ratings changes are governed via a homogeneous Markov transition matrix. Our model introduces time-varying intensities into a recursive pricing framework, allowing for the possibility of negative correlations between credit spreads and interest rates. However if a discretization is the only practical solution, it should be noted that Lando (1998) also suggests a discrete-time approximation of his solution that allows for the computation of prices without the need for the strong assumption of constant eigenvectors of the generator matrix. Thus the marginal contribution of our approach still awaits empirical verification. In the next section, we circumvent technical hurdles via additional assumptions and derive essentially closed-form solutions for the special case of step-up only bonds.

3 Step-up Only Bonds

Although most credit-sensitive bonds in circulation contain both step-up and step-down provisions, some bond issues only afford a step-up provision in the event that the issuing firm is downgraded by a ratings agency. Vorst, Houweling, and Mentink (2004) classify existing credit-sensitive bonds into categories based on the characteristics of the step-up language. For instance, a Type C bond only allows for a single step-up with an immediate adjustment to the coupon rate. This section focuses on the class of securities that allow only step-up payments. We now lay the groundwork for a model that facilitates the development of essentially closed form solutions.

We deviate from the interpretation of the generator matrix given in the previous sections and examine a set of assumptions with the purpose of obtaining a more analytically tractable model. In this setting the generator matrix does not explicitly account for the possibility of an upgrade in the credit rating of the firm, although this will be implicitly captured by the new interpretation given the intensities. For the step-up only case we modify the generator matrix as given below

$$\Lambda^c(t) = \begin{pmatrix} -\lambda^c_{1t} & \lambda^c_{1t} & 0 & 0 & \dots & 0 \\ 0 & -\lambda^c_{2t} & \lambda^c_{2t} & 0 & \dots & 0 \\ 0 & 0 & -\lambda^c_{3t} & \lambda^c_{3t} & & 0 \\ \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

This matrix only allows for a single downgrade in the credit rating at any given time. An upgrade in the credit rating does not lead to a new coupon class transition. The actual credit rating can naturally increase, however this is not explicitly modeled by the generator matrix of coupon classes.

In this setting the pricing formula simplifies and resembles the standard defaultable bond pricing equation of Lando (1998) where the stochastic recovery value is now replaced by the new price $P^{i+1}_{w,s}$

$$P^i_{t,s} = C_i E\left[e^{-\int_t^s (r_u+\lambda^c_{iu})du}|\mathcal{G}_t\right] + \int_t^s E\left[e^{-\int_t^w (r_u+\lambda^c_{iu})du}\lambda^c_{iw}P^{i+1}_{w,s}|\mathcal{G}_t\right]dw. \tag{32}$$

For applications one would like be able to show how the above matrix $\Lambda^c(t)$ collapses from the matrix $\Lambda(t)$ given in the previous section. However, this is a difficult issue to resolve. The intuition being that conditioned on a ratings upgrade, an increase in the coupon rate can only occur when the credit rating process returns to the original state and then downgrades to the trigger credit level. Therefore the intensities given by $\Lambda^c(t)$ will in general be lower than the corresponding ratings downgrade intensities in $\Lambda(t)$. A more tractable solution may involve augmenting the state vector X_t with an additional factor that measures credit level of the firm and allowing the coupon transition intensities to depend on this additional factor. For purposes of illustrating the model, we simply assume $\Lambda^c(t)$ to be an $\mathcal{F}_t$ adapted process and proceed.

3.1 Special Case of Affine Models, $K = 3$.

In this section, we assume that the underlying state vector capturing the risk factors of the firm is described by a vector $X_t \in \mathbb{R}^d$ that solves the following stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB^Q. \tag{33}$$

From this point forward we will only work with the generator matrix characterizing the transitions in coupon rates and for notational convenience will drop the c superscript and denote the intensities simply by λ_{it}. We allow λ_{it} and the spot rate r_t to depend on the underlying state process X_t in the sense that

$$\lambda_{it} = \lambda_i(X_t) = \lambda_{i0} + \lambda_{i1}\cdot X_t = \tag{34}$$

$$r_t = R(X_t) = \rho_o + \rho_1 \cdot X_t. \tag{35}$$

To facilitate analytical tractability we will now assume that μ, $\sigma\sigma^T$, λ_i and R are affine real-valued $\mathcal{F}_t$-measurable functions of the underlying state vector X_t.

In this affine setting the price of a zero-coupon bond is known to be an exponential affine function of the underlying state vector. Duffie and Kan (1996) show that if μ, $\sigma\sigma^T$ and R are affine in X_t then (under technical conditions) $E\left[e^{-\int_t^s (R(X_u))du}|\mathcal{G}_t\right]$ is exponentially affine in X_t.[11] Since λ_i is affine in X_t it follows that

$$E\left[e^{-\int_t^s (r_u+\lambda_{iu})du}|\mathcal{G}_t\right] = e^{\alpha(t,s)+\beta(t,s)\cdot X_t} \tag{36}$$

where the coefficients, $\alpha(t,s)$ and $\beta(t,s)$ are solutions to a set of Ricatti equations.

As a special case we take the situation where $K = 3$. Here the coupons depend on only 3 credit classes: investment grade ($i = 1$), speculative grade or junk ($i = 2$) and default ($i = 3$).[12] The

[11] Details can also be found in Duffie (2001), Section 7I.

[12] Investment grade bonds are defined by Moody's to have a credit rating higher than Baa. S&P defines investment grade bonds to have a rating higher than BBB. Bonds with lower credit ratings are considered speculative grade or junk bonds.

coupon payment contains a step-up only provision should the credit rating transition from investment to speculative grade. As an example, the Dutch telecom firm, KPN issued a step-up only bond allowing for only a single coupon step-up should the credit rating be downgraded to a trigger level. At least in theory, such securities are attractive for pricing purposes in that an essentially closed form solution can be obtained.

Recalling that the generator matrix characterizes transitions in the coupon rate and not explicit transitions in the credit class, we have

$$\Lambda^c(t) = \begin{pmatrix} -\lambda_{1t} & \lambda_{1t} & 0 \\ 0 & -\lambda_{2t} & \lambda_{2t} \\ 0 & 0 & 0 \end{pmatrix},$$

.

We now derive a pricing formula for bonds characterized by this generator matrix. First, using the Duffie and Kan (1996) result, the value of a coupon payment on a speculative grade ($i = 2$) bond given our no recovery assumption can be written as

$$P^2_{t,s} = E\left[e^{-\int_t^s (r_u+\lambda_{iu})du} C_2 | \mathcal{G}_t\right] = C_2 e^{\alpha(t,s)+\beta(t,s)\cdot X_t} \tag{37}$$

Next, for an investment grade bond ($i = 1$) the value of a coupon payment is given by

$$P^1_{t,s} = C_1 E\left[e^{-\int_t^s (r_u+\lambda_{1u})du} | \mathcal{G}_t\right] + \int_t^s E\left[e^{-\int_t^w (r_u+\lambda_{1u})du} \lambda_{1w} P^2_{w,s} | \mathcal{G}_t\right] dw \tag{38}$$

$$= C_1 e^{\alpha(t,s)+\beta(t,s)\cdot X_t} + C_2 \int_t^s E\left[e^{-\int_t^w (r_u+\lambda_{1u})du} \lambda_{1w} e^{\alpha(w,s)+\beta(w,s)\cdot X_w} | \mathcal{G}_t\right] dw. \tag{39}$$

Since λ_{1w} is an affine function and $P^2_{w,s}$ an exponentially affine function of the underlying state vector, under technical regularity conditions we can, further simplify the above by using the extended transform of Duffie, Pan, and Singleton (2000) Section 2.3, which shows that

$$E\left[e^{-\int_t^s R(X_u)du} (\nu_0 + \nu_1 \cdot X_s) e^{\mu \cdot X_s} | G_t\right] = e^{a(t,s)+b(t,s)\cdot X_t} (A(t,s) + B(t,s) \cdot X_t)$$

where A and B satisfy ordinary differential equations. Hence, the expectation in the integrand of equation (39) can be eliminated and the computation of the price now only requires that a numerical integral be performed

$$P^1_{t,s} = C_1 e^{\alpha(t,s)+\beta(t,s)\cdot X_t} + C_2 \int_t^s e^{a(t,w)+b(t,w)\cdot X_t} (A(t,w) + B(t,w) \cdot X_t) dw. \tag{40}$$

Proposition 2 *Consider a credit-sensitive coupon vector $C = (C_1, C_2, 0)^T$ to be paid at time s and let C_i denote the ith element of C. Suppose the initial rating of the firm is i at time t and the credit rating process is characterized by the generator matrix $\Lambda^c(t)$. Under technical conditions and assuming zero recovery upon default, the value of a step-up only coupon payment is given by (37) and (40).*

3.2 Special Case of Affine Models, $K = 4$

In this section we extend our results to a setting with 4 credit classes: A-grade (i=1), B-grade (i=2), speculative grade or junk (i=3) and default (i=4).

We maintain all the affine assumptions given in the previous section. By shifting the credit class indices on the equations derived in the previous section, we see that the pricing equations for both B-grade (i=2) and junk (i=3) bonds are given by

$$P^3_{t,s} = C_2 e^{\alpha(t,s)+\beta(t,s)\cdot X_t} \tag{41}$$

$$P^2_{t,s} = C_2 e^{\alpha(t,s)+\beta(t,s)\cdot X_t} + C_3 \int_t^s \Phi^2(t,v)dv \tag{42}$$

where

$$\Phi^2(t,v) = e^{a(t,v)+b(t,v)\cdot X_t}(A(t,v) + B(t,v)\cdot X_t).$$

It only remains to obtain an expression for the highest credit class (i =1). We begin with

$$P^1_{t,s} = C_1 e^{\alpha(t,s)+\beta(t,s)\cdot X_t} + \int_t^s E\left[e^{-\int_t^w (r_u+\lambda_{1u})du}\lambda_{1w}P^2_{w,s}|\mathcal{G}_t\right]dw. \tag{43}$$

and by substituting the expression for $P^2_{w,s}$ into the integrand of the second term obtain

$$\int_t^s E\left[e^{-\int_t^w (r_u+\lambda_{1u})du}\lambda_{1w}P^2_{w,s}|\mathcal{G}_t\right]dw \tag{44}$$

$$= C_2 \int_t^s E\left[e^{-\int_t^w (r_u+\lambda_{1u})du}\lambda_{1w}e^{\alpha(w,s)+\beta(w,s)\cdot X_w}|\mathcal{G}_t\right]dw \tag{45}$$

$$+ C_3 \int_t^s E\left[e^{-\int_t^w (r_u+\lambda_{1u})du}\lambda_{1w}\int_w^s \Phi^2(w,v)dv|\mathcal{G}_t\right]dw.$$

An application of the Duffie, Pan, and Singleton (2000) extended transform result to the first term in (45) yields

$$C_2 \int_t^s E\left[e^{-\int_t^w (r_u+\lambda_{1u})du}\lambda_{1w}e^{\alpha(w,s)+\beta(w,s)\cdot X_w}|\mathcal{G}_t\right]dw = C_2\int_t^s \Phi^1(t,w,s)dw \tag{46}$$

where

$$\Phi^1(t,w,s) = e^{a(t,w,s)+b(t,w,s)\cdot X_t}(A(t,w,s) + B(t,w,s)\cdot X_t).$$

The second term in (45) can be rewritten as

$$C_3 \int_t^s E\left[e^{-\int_t^w (r_u+\lambda_{1u})du}\lambda_{1w}\int_w^s \Phi^2(w,v)dv|\mathcal{G}_t\right]dw \tag{47}$$

$$= C_3 \int_t^s \int_w^s E\left[e^{-\int_t^w (r_u+\lambda_{1u})du}\lambda_{1w}\Phi^2(w,v)|\mathcal{G}_t\right]dvdw \tag{48}$$

$$= C_3 \int_t^s \int_w^s E\left[e^{-\int_t^w (r_u+\lambda_{1u})du}\lambda_{1w}e^{\alpha(w,v)+\beta(w,v)\cdot X_w}A(w,v)|\mathcal{G}_t\right] \tag{49}$$

$$+ E\left[e^{-\int_t^w (r_u+\lambda_{1u})du}\lambda_{1w}e^{\alpha(w,v)+\beta(w,v)\cdot X_w}B(w,v)\cdot X_w)|\mathcal{G}_t\right]dvdw$$

The first term in (49) simplifies from another application of Duffie, Pan, and Singleton (2000). The second term in (49) contains the the product of a quadratic (the product of 2 affine functions) and an exponential affine function. We obtain

$$C_3 \int_t^s E\left[e^{-\int_t^w (r_u+\lambda_{1u})du}\lambda_{1w}\int_w^s \Phi^2(w,v)dv|\mathcal{G}_t\right]dw = C_3\int_t^s\int_w^s \Phi(t,w,v) + \Psi(t,w,v)dvdw. \tag{50}$$

where

$$\Phi(t,w,v) = e^{\gamma(t,w,v)+\delta(t,w,v)\cdot X_t}(\Gamma(t,w,v) + \Delta(t,w,v)\cdot X_t) \tag{51}$$
$$\Psi(t,w,v) = e^{c(t,w,v)+d(t,w,v)\cdot X_t}\left(C(t,w,v) + D(t,w,v)\cdot X_t + X_t' E(t,w,v)\cdot X_t\right). \tag{52}$$

This result follows from an extension of the extended transform - that the expected discounted value of the product of a quadratic and an exponential affine can be written as Ψ.[13] Combining our results we obtain

$$P^1_{t,s} = C_1 e^{\alpha(t,s)+\beta(t,s)\cdot X_t} + C_2 \int_t^s \Phi^1(t,w,s)dw + C_3 \int_t^s \int_w^s \Phi(t,w,v) + \Psi(t,w,v)dvdw \tag{53}$$

Proposition 3 *Consider a credit-sensitive coupon vector $C = (C_1, C_2, C_3, 0)^T$ to be paid at time s and let C_i denote the ith element of C. Suppose the initial rating of the firm is i at time t and the credit rating process is characterized by the generator matrix $\Lambda^c(t)$. Under technical conditions and assuming zero recovery upon default, the value of a step-up only coupon payment is given by (41), (42) and (53).*

4 Conclusion

Building on the work of Lando (1998), we develop a general framework for pricing ratings-sensitive variable coupon bonds where the ratings transitions are modeled via a inhomogeneous continuous-time Markov chain. This paper adds a new dimension to the literature by introducing an alternative set of models for pricing these instruments within an arbitrage-free setting. For the case of a bond with both step-up and step-down provisions, we develop a general continuous-time recursive pricing framework. For the specific case of step-up only bonds, we derive essentially closed-form solutions for bonds with 3 and 4 credit classes under additional affine assumptions. The continuous-time framework provides a more realistic modeling of ratings migrations than discrete-time model counterparts found in the literature. The modeling of state-dependent time-varying transition intensities in an affine term structure framework allows for correlation between credit spreads and the short rate. Empirical research is necessary to establish the relevance of these models for use in applications.

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[13] Proof required! To be provided.

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