What is the Price of Interest Risk in the Brazilian Swap Market?

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Abstract

In this paper, we adopt a polynomial arbitrage-free dynamic term structure model to analyze the risk premium structure of the main Brazilian interest rate market. Closed-form bond pricing formulas provide a clear interpretation of each source of aggregate risk in this economy since risk factors are term structure movements predetermined by the model. The parametric nature of our model, allows for an important approximation of the time series of risk factors by running a sequential set of linear regressions independent across time. We use this approximation to provide a simple method to extract the risk premium embeded in interest rate zero coupon instruments without having to run a full optimization of a dynamic model. We apply this methodology considering three sourcers of aggregate risk (level, slope and curvature) on a comprehensive dataset of interest rate swaps and identify that the three movements are priced by investors in this market. Our new methodology should be especially useful for market practitioneers in risk management, economic analysis of the term structure and option pricing.

Keywords: Term structure of interest rates, parametric models, affine models, cross sectional estimation, time series analysis.

JEL Classification: C1,C5,G1

1 Introduction

Parametric arbitrage-free term structure models (PAFTSMs) are dynamic models presenting analytical closed-form bond pricing formulas explicitly depending on the risk factors underlying the economy. Although theoretical results on such models have been derived by a large set of papers[1], corresponding empirical implementations have only recently gained more attention with the work of Almeida and Vicente (2008) and Christensen, Diebold and Rudebusch (2011). Although those two papers test for the importance of no-arbitrage on forecasting, the main message behind their methodology is that parametric models are *useful and simpler to understand* because their implied term structure movements are known before model estimation.[2]

Knowing the term structure movements or latent factors before estimating the model brings a number of advantages when compared to general Dynamic Term Structure Models (see Duffie and Kan (1996), Singleton (2008)). First, closed-form formulas for bond prices guarantee that each source of risk in the economy (or dynamic factor) has a direct interpretation as an specific movement of the term structure. Consequently, risk premia is directly associated with well defined yield movements. Second, understanding practical implications of these models might improve hedging errors when compared to dynamic models of the Heath Jarrow Morton (1992) type with sequential inconsistent recalibration[3]. Third and most importantly, the a priori knowledge of approximate latent factors allows for a better understanding of the role of model parameters on simultaneously capturing cross-sectional and dynamics information on bond prices/yields[4].

In this paper we fully explore this last point to produce a new, simple, and much faster method to estimate the risk premia implicit in a fixed income market without having to run full-optimization of the dynamic term structure model.

In order to obtain our results we implement the Legendre dynamic model, a separable term structure model (Filipovic (2002)) which parameterizes the term structure of interest rates as a linear combination of Legendre polynomials, where coefficients of the combination represent the state vector. Almeida (2005) presents stochastic differential equations (SDEs) for the state variables which turn the model into a parametric arbitrage-free term structure

[1] Bjork and Christensen (1999), Filipovic (2000), Bjork and Landen (2002), Filipovic (1999), Filipovic (2000b), Filipovic (2002), Sharef and Filipovic (2004) and Almeida (2005).

[2] Those papers show that imposing no-arbitrage on parametric term-structure models reduces overfitting by limiting the number of free parameters and improve their forecasting ability in out-of sample forecasting.

[3] According to Buraschi and Corrieli (2003), inconsistent recalibration means estimating a cross-sectional yield curve based on an statistical model which generates a curve outside the family of curves generated by the chosen dynamic HJM-type model.

[4] For a general discussion on efficient estimation of Gaussian affine term structure models see Joslin, Singleton and Zhu (2011).

model. The dynamics of the state vector, after the imposition of no-arbitrage restrictions, is more general than the dynamics of traditional affine models. The generality appears on the factors' diffusion coefficients that can be represented by general continuous functions of state variables satisfying integrability conditions, as opposed to the affine case, where they are represented by the square root of affine functions of state variables. Moreover, when the diffusion matrix is restricted to be an affine function of the dynamic factors, the model becomes an example of affine model with extra conditionally deterministic factors (see Almeida and Vicente (2008))[5].

We explore the closed-form formulas for bond prices to partially separate the estimation procedure in two parts. In the first step, we invert the state vector from fixed income prices only fixing few volatility model parameters (connected to the conditionally deterministic factors). In the second step, after having extracted the state space vector, we concentrate on the estimation of the other parameters based on the dynamics imposed by the SDEs, using maximum likelihood estimation.

The only reason for the separation of the estimation process in two stages not to work perfectly is the existence of the above mentioned conditionally deterministic factors. However, Almeida and Vicente (2008) have shown, based on U.S interest rate data, that under the Legendre arbitrage-free polynomial model, the difference between the implied latent factors running the full dynamic model and the implied latent factors obtained by running cross-sectional regressions is very small, of the order of 20 basis points at most.

We build on the work of Almeida and Vicente (2008) to obtain our simple method for risk premia extraction. First, we confirm the results of negligibility of the difference between the state vector implied by the fully optimized dynamic model and the cross-sectional obtained by Almeida and Vicente (2008), but for an specific long Brazilian dataset of interest rate swaps. Moreover, we show that for this dataset the difference is even smaller, of less than 8 basis points for the Gaussian model. Based on this small difference, we suggest discarding the deterministic factors and perfectly separating risk-neutral and physical estimation processes.

The separation of estimation in two stages for the Gaussian polynomial model allows us to come up with our new method for risk premia estimation. More specifically, we show that risk premia can be obtained by first running one multiple regression of observed yields on the parametric polynomial Legendre factors and, after obtaining the time-series of the latent factors, running one continuous-time Auto-regression of order one on the time series of the latent factors to obtain parameters of the physical probability measure. Risk premia

[5] Trolle and Schwartz (2009) also make use of locally deterministic factors in Heath Jarrow Morton models with stochastic volatility.

is then backed out from a simple analytical formula that connects risk neutral and physical probability measures. This can be done in less than a second, and most importantly it does not require any initial values for the parameters of the dynamic model.

Our new method offers the possibility of testing what type of movements investors care about. We apply the methodology to the swaps database adopting a three-factor arbitrage-free Legendre polynomial model and end up finding that the level, the slope and the curvature of the term structure all drive the risk premia of the Brazilian term structure. Slope is the main source of risk driving level risk premia, a result consistent with Fama and Bliss (1987), which document strong dependence between bond excess returns and slope of the term structure. We also have a new result on curvature, which is priced in this market, while it is usually only useful in forecasting analysis, but not priced. We attribute this result to the complex dynamic structure of the Brazilian swap market, which, according to a principal component analysis demands four factors to fully capture its variability across time.

The rest of the paper is organized as follows. Section 2 presents the cross sectional Legendre model, and the dynamic Legendre model. In Section 3, first the adopted data set is introduced, and then the Gaussian multi-factor version of the model is implemented. Section 4 concludes. Appendix I presents a set of important rotation matrices necessary when implementing the Legendre model.

2 The Legendre Polynomial Model

2.1 The Static Model

Almeida et al. (1998) proposed modeling the term structure of interest rates in Emerging Eurobeach markets as a benchmark curve plus a linear combination of Legendre polynomials:

$$R(\tau) = B(\tau) + \sum_{n\geq 0} c_n P_n(\frac{2\tau}{l} - 1) \qquad (1)$$

where τ denotes time to maturity, $B(\tau)$ the benchmark curve, $P_n(.)$ is the Legendre polynomial of degree n, and l is the longest maturity of a bond in the market considered. The first four Legendre polynomials are respectively 1, x, $\frac{1}{2}(3x^2 - 1)$, and $\frac{1}{2}(5x^3 - 3x)$, and usually they are enough to capture movements in applications with real data (see Almeida et al. (2003) for an application using Brazilian Sovereign bonds). Figure 1 depicts the first four Legendre polynomials. They have a nice interpretation in terms of types of term structure movements that each one drives. The constant polynomial is related to parallel shifts; the linear polynomial is related to changes in the slope; the quadratic polynomial is related to changes in the curvature; the cubic polynomial is related to a more complex change in the curvature.

In the estimation process, the number of Legendre polynomials is limited, according to information criteria tests, say to a total of N. In the case of Eurobeach markets, and/or Bray Bond markets, there is no observable term structure and the set of parameters $\theta = (c_0, c_1, ..., c_N)'$ is obtained solving the non-linear problem of minimizing the distance between the model implied prices and the observed bond prices. The minimization is implemented by discounting the cash flows of these bonds using the term structure parameterized in Equation (1). For observable term structures, such as the Brazilian term structure of swaps fixed-floating rates, the model can be estimated through a linear regression, as shown in what follows. Assume that we observe the term structure $\tilde{R}(\tau)$ with measurement error and $B(\tau)$ without measurement error, for the maturities $m_1, m_2, ..., m_k$.

Define $y = \begin{bmatrix} \tilde{R}(m_1) - B(m_1) \\ \tilde{R}(m_2) - B(m_2) \\ \vdots \\ \tilde{R}(m_k) - B(m_k) \end{bmatrix}$, and $L = \begin{bmatrix} P_0(m_1) & P_1(m_1) & ... & P_{N-1}(m_1) \\ P_0(m_2) & P_1(m_2) & ... & P_{N-1}(m_2) \\ \vdots & \vdots & & \vdots \\ P_0(m_k) & P_1(m_k) & ... & P_{N-1}(m_k) \end{bmatrix}$.

The parametrization given in Equation (1) implies the regression:

$$y = L\theta + \epsilon \tag{2}$$

whose solution is known in closed-form:

$$\hat{\theta} = (L'L)^{-1}L'y \tag{3}$$

2.1.1 Legendre Forward Rates

We are interested in obtaining the instantaneous forward rate curve implied by the Legendre model because it will be useful in the obtention of the dynamic model.

The relation between the instantaneous forward rate curve and the term structure of interest rates is given by:

$$r(t, \tau) = R(t, \tau) + \tau \frac{\partial R(t, \tau)}{\partial d\tau} \tag{4}$$

An application of Equation (4) to the Legendre parameterized term structure appearing in Equation (1) yields:

$$r(\tau) = B(\tau) + \tau \frac{\partial B(\tau)}{\partial \tau} + \sum_{n \geq 0} c_n P_n(\frac{2\tau}{l} - 1) + \tau \left(\sum_{n \geq 1} c_n \frac{\partial P_n(\frac{2\tau}{l} - 1)}{\partial \tau} \right) \tag{5}$$

Suppose we are interested in obtaining the forward curve for the Legendre model with the first four Legendre polynomials, considering a null benchmark curve. Defining the auxiliary variable $x = \frac{2\tau}{l} - 1$, and using the chain rule to get $\frac{\partial P_n(\frac{2\tau}{l}-1)}{\partial \tau} = \frac{2}{l}\frac{\partial P_n(x)}{\partial x}$, Equation (5) becomes:

$$r(\tau) = c_0 + c_1 x + \frac{c_2}{2}(3x^2 - 1) + \frac{c_3}{2}(5x^3 - 3x) + \frac{2\tau}{l}\left[c_1 + 3c_2 x + \frac{c_3}{2}(15x^2 - 3)\right] \tag{6}$$

Observing Equation (6), and fixing the number of Legendre polynomials to be N, note that the instantaneous forward rate can be rewritten as a polynomial in the maturity variable τ:

$$r(\tau) = \sum_{n=0}^{N-1} L_n(c_0, c_1, ..., c_{N-1})\tau^n \tag{7}$$

where each L_n is a linear function. In the previous example of Equation (6), $L_0 = c_0 - c_1 + c_2 - c_3$, $L_1 = \frac{4}{l}c_1 - \frac{12}{l}c_2 - \frac{51}{l}c_3$, and so on. As shown later, this way of writing the forward rate equation will be very important to identify the state space of Legendre dynamic models.

2.2 The Dynamic Model

Duffie and Kan (1996) define affine term structure models as the ones whose short term rate and bond prices are respectively affine and exponential-affine functions of the dynamic factors. Assuming the existence of an equivalent martingale measure $\mathcal{Q}$[6], they show that any multi-dimensional affine term structure model, under some regularity conditions, can be represented by a stochastic differential system whose drift and squared diffusion matrix are affine under $\mathcal{Q}$:

$$dY_t = \mu^{\mathcal{Q}}(Y_t)dt + \sigma(Y_t)dW_t^* \tag{8}$$

where W_t^* is an N-dimensional Brownian Motion under $\mathcal{Q}$, $\mu^{\mathcal{Q}}(Y_t) = \kappa^{\mathcal{Q}}(\theta^{\mathcal{Q}} - Y_t)$ and $\sigma(Y_t) = \Sigma\sqrt{S_t}$, $S_t^{ii} = \alpha_i + \beta_i^T Y_t$, $S_t^{ij} = 0, i \neq j$, $1 \leq i, j \leq n$, $\alpha_i \in \mathcal{R}, \beta_i \in \mathcal{R}^N$. They further obtain a closed-form formula (up to solving the two ODEs presented bellow) for the time t price of a zero coupon bond with time of maturity T, $Z(t,T)$:

$$Z(t,T) = e^{A(\tau)+B(\tau)'Y_t}, \tag{9}$$

[6]Under this measure, also known as pricing measure, discounted prices of bonds are martingales, what guarantees no-arbitrage in the market.

where $\tau = T - t$ and $A(.)$ and $B(.)$ are solutions of the following Ricatti ODEs:

$$B'(t) = \rho_1 - \kappa^{\mathcal{Q}}.B(t) - \tfrac{1}{2}B(t)^T H_1 B(t), \quad B(T) = 0 \tag{10}$$

$$A'(t) = \rho_0 - \kappa^{\mathcal{Q}}\theta^{\mathcal{Q}}.B(t) - \tfrac{1}{2}B(t)^T H_0 B(t), \quad A(T) = 0 \tag{11}$$

where $\sigma(Y_t)\sigma(Y_t)^T = \Sigma S_t \Sigma^T = H_0 + H_1 Y_t$ and ρ_0 and ρ_1 come from the definition of the short term rate $r_t = \rho_0 + \rho_1 Y_t$.

Whenever using maximum likelihood estimation methods, these ODEs should be solved on each optimization passage. Although they are not computationally costly, it would be much nicer if we did not have to worry about them. That is the case with parametric term structure models!

In clear contrast with the Dynamic Affine models proposed by Duffie and Kan (1996), in the dynamic version of the Legendre model bond prices are given by the following equation:

$$P(t,T) = e^{-\tau G(\tau)' Y_t}, \tag{12}$$

where $G(\tau)$ is a vector containing the first N Legendre polynomials evaluated at maturity τ:

$$G(\tau) = [P_0(\frac{2\tau}{l} - 1)\ P_1(\frac{2\tau}{l} - 1)\ ...\ P_{N-1}(\frac{2\tau}{l} - 1)] \tag{13}$$

The previous two equations clearly show that there is no necessity of solving ODEs in order to be able to obtain the price of a zero coupon bond under the Legendre model. Of course, as "There is no free lunch", in order to be able to avoid the Ricatti equations there is a lost in model flexibility. In fact, imposing specific forms for the maturity functions A and B restricts the stochastic differential system satisfied by the dynamic factors. In a N-dimensional Brownian setting, using Filipovic's (1999) results, Almeida (2005) showed that a Dynamic Legendre Polynomial Model with N factors presents only $[\frac{N}{2}]$ risky factors, that is only $[\frac{N}{2}]$ elements in the Brownian vector drive uncertainty of the term structure.

For practical purposes Almeida and Vicente (2008) showed that the fact that the dynamic Legendre model restricts the Brownian structure does not affect the performance of the model. They show that if one wants to have uncertainty driven by a k dimensional Brownian Motion it is enough to model the term structure as a linear combination of $2k$ Legendre polynomials, where some of the factors will be locally deterministic, in a perfect accordance with both the general formulation of affine models which appears in Collin Dufresne et al. (2009), as well as with the formulation of affine models with lagged variables presented in Joslin et al. (2004).

So, if we want to model the term structure dynamics with three risky factors driving its uncertainty, as Litterman and Scheinkman (1991) showed to be statistically adequate to model the evolution of the U.S. Term Structure of Treasury bonds[7], we would need a Legendre dynamic model with six factors. Almeida and Vicente (2008) have constructed such six-factor models under both Gaussian and Stochastic Volatility environments.

In this paper we make use of the restrictions imposed to risk-neutral parameters to show that in the Gaussian model the conditionally deterministic factors are so small that we can discard them. Once we can discard them, risk premia can be approximated by a very simple procedure described below.

2.2.1 The Gaussian Six-Legendre Polynomial Dynamic Model

From Equations (12) and (13), it is straightforward to extract the term structure of interest rates as a function of the state vector under the Dynamic Legendre Model:

$$R(t,\tau) = G(\tau)'Y_t = \sum_{n=1}^{N} Y_{t,n} P_{n-1}(\frac{2\tau}{l} - 1) \quad (14)$$

Almeida (2005) shows that in order to impose model restrictions that preclude arbitrages, it is convenient to use an auxiliary state process $\tilde{Y}$, which is related to the forward rates curve by the following equation:

$$r(t,\tau) = \sum_{n=1}^{N} \tilde{Y}_{t,n} \tau^{n-1} \quad (15)$$

where $\tilde{Y}$ dynamics is given by[8]:

$$d\tilde{Y}_t = \tilde{\mu}^{\mathcal{Q}}(\tilde{Y}_t)dt + \tilde{\sigma}(\tilde{Y}_t)dW_t^* \quad (16)$$

Almeida (2005) proves that discounted bond prices are $\mathcal{Q}$-martingales whenever the following restriction holds[9] for the drift $\tilde{\mu}^{\mathcal{Q}}(\tilde{Y}_t)$:

$$\sum_{j=2}^{N} (j-1)\tilde{Y}_{t,j}\tau^{j-2} = \sum_{j=1}^{N} (\mu^{\mathcal{Q}}(\tilde{Y}_t))_j \tau^{j-1} - \sum_{j=1}^{[\frac{N}{2}]} \sum_{k=1}^{[\frac{N}{2}]} \tilde{\Gamma}_{jk}(\tilde{Y}_t) \frac{\tau^{j+k-1}}{k} \quad (17)$$

with $\tilde{\Gamma}(\tilde{Y}_t) = \sigma(\tilde{Y}_t)\sigma(\tilde{Y}_t)^T$. This equation demonstrates that although the prices of bonds

[7] More recently, Heidari and Wu (2003) obtain empirical results showing that three factors are also statistically adequate to explain the dynamics of the U.S. term structure of swap yields.

[8] By the fact that $\tilde{Y}$ is obtained through a linear transformation of Y, whenever Y dynamics is affine $\tilde{Y}$ dynamics will also be affine.

[9] Despite the consistency restriction being expressed in terms of the drift of the auxiliary state space vector $\tilde{Y}$, one specific affine transformation allows us to express it using the drift of the original state process Y. An example follows soon.

on the Legendre model are affine functions of the state space vector Y, the dynamics of Y (which is a linear transformation of $\tilde{Y}$) is not restricted to be affine[10].

Whenever we restrict the diffusion coefficient of the state vector to present the same functional form of the Gaussian affine class, the SDE for $\tilde{Y}$ becomes:

$$d\tilde{Y}_t = \tilde{\mu}^{\mathcal{Q}}(\tilde{Y}_t)dt + \tilde{\Sigma}dW_t^* \tag{18}$$

and consequently the consistency restriction becomes:

$$\sum_{j=2}^{N}(j-1)\tilde{Y}_{t,j}\tau^{j-2} = \sum_{j=1}^{N}(\mu^{\mathcal{Q}}(\tilde{Y}_t))_j\tau^{j-1} - \sum_{j=1}^{[\frac{N}{2}]}\sum_{k=1}^{[\frac{N}{2}]}\tilde{H}_{0,jk}\frac{\tau^{j+k-1}}{k} \tag{19}$$

with $\tilde{\Sigma}\tilde{\Sigma}^T = \tilde{H}_0$.

The original state space vector Y can be obtained from $\tilde{Y}$ by solving the following linear system:

$$\tilde{Y}_{t,j} = L_j(Y_t), j = 1, 2, ..., N. \tag{20}$$

with $L_j, j = 1, 2, ..., N$ coming from Equation (7).

By matching coefficients on the time to maturity variable τ in Equation (19), we obtain an explicit expression for the drift[11]:

$$\begin{aligned}
&\tilde{\mu}^{\mathcal{Q}}(\tilde{Y}_t)_{2j-1} = (2j-1)\tilde{Y}_{t,2j} + \tfrac{\tilde{H}_{0,(j-1)j}}{j} + \tfrac{\tilde{H}_{0,j(j-1)}}{j-1} +,\ 1 \leq j \leq [\tfrac{N}{2}]\\
&\tilde{\mu}^{\mathcal{Q}}(\tilde{Y}_t)_{2j} = 2j\tilde{Y}_{t,2j+1} + \tfrac{\tilde{H}_{0,jj}}{j} + \tfrac{\tilde{H}_{0,(j+1)(j-1)}}{j-1},\ 1 \leq j \leq [\tfrac{N-1}{2}]\\
&\tilde{\mu}^{\mathcal{Q}}(\tilde{Y}_t)_N = \left\{ \begin{array}{ll} \frac{4(N-2)}{(N-1)(N-3)}\tilde{H}_{0,\frac{N-3}{2}\frac{N-1}{2}} & ,\text{N odd} \\ \frac{2}{N}\tilde{H}_{0,\frac{N}{2}\frac{N}{2}} & ,\text{N even} \end{array} \right\}
\end{aligned} \tag{21}$$

where we assume null values for all terms where at least one index for matrix $\tilde{H}_0$ is zero.

Using Equations (8) and (18) to specify a Six-Legendre dynamic model, and Equation

[10]For instance, if we choose the functional form of $\tilde{\sigma}(\tilde{Y})$ to be different of an square root of an affine function of $\tilde{Y}$, the dynamics of both $\tilde{Y}$ and Y will be non-affine.

[11]When constructing the proof of consistency of the Legendre Dynamic Model, Almeida (2005) proposes arbitrage-free examples of term structure models parameterized by Legendre polynomials. However, he does not provide as general models as provided here with Equation (21).

(21) to enforce the drift restriction we have:

$$
\begin{array}{l}
\tilde{\Sigma}^{ii} = \left\{ \begin{array}{cc} \tilde{\alpha}_i & , i \leq [\frac{N}{2}] \\ 0 & , i > [\frac{N}{2}] \end{array} \right\} \\
\tilde{\Sigma}_{i,j} = 0, i, j > [\frac{N}{2}] \\
\tilde{\mu}^{\mathcal{Q}}(\tilde{Y}_t)_1 = \tilde{Y}_{t,2} \\
\tilde{\mu}^{\mathcal{Q}}(\tilde{Y}_t)_2 = 2\tilde{Y}_{t,3} + \tilde{H}_{0,11} \\
\tilde{\mu}^{\mathcal{Q}}(\tilde{Y}_t)_3 = 3\tilde{Y}_{t,4} + \frac{\tilde{H}_{0,12}}{2} + \tilde{H}_{0,21} \\
\tilde{\mu}^{\mathcal{Q}}(\tilde{Y}_t)_4 = 4\tilde{Y}_{t,5} + \frac{\tilde{H}_{0,22}}{2} + \tilde{H}_{0,31} \\
\tilde{\mu}^{\mathcal{Q}}(\tilde{Y}_t)_5 = 5\tilde{Y}_{t,6} + \frac{\tilde{H}_{0,23}}{3} + \frac{\tilde{H}_{0,32}}{2} \\
\tilde{\mu}^{\mathcal{Q}}(\tilde{Y}_t)_6 = \frac{\tilde{H}_{0,33}}{3}
\end{array} \tag{22}
$$

In order to obtain the original state space dynamics we have to solve for Equation (20). To that end, we also have to know the parametric forms of the Legendre polynomials of degrees four and five:

$$
\begin{array}{l}
P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \\
P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)
\end{array} \tag{23}
$$

Performing a calculation very similar to the one obtained in subsection 2.1.1, assuming $x = \frac{2\tau}{l} - 1$, we obtain the forward curve under this model:

$$
\begin{array}{l}
r(t,\tau) = Y_{t,1} + Y_{t,2}x + \frac{Y_{t,3}}{2}(3x^2 - 1) + \frac{Y_{t,4}}{2}(5x^3 - 3x) + \frac{Y_{t,5}}{8}(35x^4 - 30x^2 + 3) + \\
+ \frac{Y_{t,6}}{8}(63x^5 - 70x^3 + 15x) + \frac{2\tau}{l}\left[Y_{t,2} + 3Y_{t,3}x + \frac{Y_{t,4}}{2}(15x^2 - 3)\right] + \\
+ \frac{2\tau}{l}\left[\frac{Y_{t,5}}{8}(140x^3 - 60x) + \frac{Y_{t,6}}{8}(315x^4 - 210x^2 + 15)\right]
\end{array} \tag{24}
$$

After some tedious algebraic manipulations (basically expressing $r(t, \tau)$ explicitly as a polynomial of degree five in the variable τ and calculating each coefficient that multiply each power in τ) we obtain:

$$
\tilde{Y}_t = LY_t \tag{25}
$$

where L is an upper triangular matrix presented in Equation (46) on Appendix I.
Note that we are ready to express the drift and diffusion restrictions for the original state vector Y, by using Equation (25). To that end, let us rewrite the drift $\tilde{\mu}^{\mathcal{Q}}$ in matrix notation, as an affine transformation of $\tilde{Y}$:

$$
\tilde{\mu}^{\mathcal{Q}}(\tilde{Y}_t) = M + U\tilde{Y}_t \tag{26}
$$

where U and M are defined by Equations (47) and (48) on Appendix I.
Finally, the drift and diffusion of process Y are given by:

$$
\mu^{\mathcal{Q}}(Y_t) = L^{-1}\tilde{\mu}^{\mathcal{Q}}(\tilde{Y}_t) = L^{-1}\tilde{\mu}^{\mathcal{Q}}(LY_t) = L^{-1}M + L^{-1}ULY_t \tag{27}
$$

$$\sigma(Y_t) = L^{-1}\tilde{\Sigma} \tag{28}$$

2.2.2 Time Varying Risk Premia

In this section we describe the element that connects the dynamics of the Legendre factors under the risk neutral (or pricing) measure $\mathcal{Q}$ and the physical measure, which we denominate P. That element is precisely the market price of risk, or in technical terms, the instantaneous volatility of the state price deflator[12]. Duffee (2002) and Dai and Singleton (2002) (henceforth DS 2002) tested the estimation of different specifications of Affine processes using historical data on U.S. treasury bonds and concluded that a flexible time varying risk premia (with market price of risk possibly depending on the different risk factors driving the movements of the term structure) is fundamental to capture the dynamics. Almeida (2004) confirms the need for time-varying risk premia when modeling the Brazilian Swap curve. Based on these results, we adopt for the structure of the risk premia the same structure adopted by Duffee (2002) in his Essentially Affine processes, directly implying the following parametric form for the market prices of risk:

$$\Lambda_t = \lambda_0 + \lambda_Y Y_t, \tag{29}$$

where λ_0 is a N x 1 vector, λ_Y is a N x N matrix.

The market prices of risk allows us to relate the two Brownian Motion vectors under both probability measures P and $\mathcal{Q}$:

$$W_t^* = W_t + \int_0^t \Lambda_s ds \tag{30}$$

where W_t is N-dimensional Brownian Motion under P.
Moreover, if the state vector presents its risk neutral dynamics expressed by:

$$dY_t = \kappa^{\mathcal{Q}}(\theta^{\mathcal{Q}} - Y_t)dt + \sigma(Y_t)dW_t^* \tag{31}$$

and its physical dynamics expressed by:

$$dY_t = \kappa(\theta - Y_t)dt + \sigma(Y_t)dW_t \tag{32}$$

a direct substitution of Equation (29) in Equation (30) and of this equation in Equation (32) yields:

$$\kappa = \kappa^{\mathcal{Q}} + \Sigma\Lambda_Y \tag{33}$$

[12]The state price deflator is a strictly positive process with the property that any price process, deflated by it, is a martingale under the physical measure. Absence of arbitrage implies the existence of at least one state price deflator (see Duffie (2001) for details).

$$\kappa\theta = \kappa^{\mathbb{Q}}\theta^{\mathbb{Q}} + \Sigma \begin{bmatrix} \lambda_0^1 \alpha_1 \\ \vdots \\ \lambda_0^N \alpha_N \end{bmatrix} \tag{34}$$

Note by Equation (27) the correspondence between $\kappa^{\mathbb{Q}}, \theta^{\mathbb{Q}}$ and matrices L, M, U:

$$\begin{aligned} \kappa^{\mathbb{Q}} &= L^{-1}UL \\ \kappa^{\mathbb{Q}}\theta^{\mathbb{Q}} &= L^{-1}M \end{aligned} \tag{35}$$

Equation (35) represents the restriction which the risk neutral drift should satisfy to avoid arbitrages: By Equations (47) and (48) it is clear that $\kappa^{\mathbb{Q}}$ and $\theta^{\mathbb{Q}}$ are specific functions of the volatility process S_t.

2.2.3 The Estimation Method

During the estimation process, knowledge of the factors dynamics under both measures is necessary, because the pricing measure is used to fit model implied yields to the observed yields, while the physical measure is used to define the conditional probability distributions of the factors, which serve as inputs for the estimation technique, usually Maximum Likelihood (Chen and Scott (1993), Duffie and Singleton (1997) and DS (2002)), Quasi Maximum Likelihood (Duffee (2002), Duarte (2004)), or Kalman Filtering (Duan and Simonato (1999), Duffee and Stanton (2004)). In particular, in this paper we follow Chen and Scott (1993) in the use of maximum likelihood estimation.

3 Empirical Results

In this section we implement the Six-Legendre Dynamic Model using Brazilian Swaps data, and the Gaussian version of the model. As will be shown on subsection 3.1, the dynamics of Brazilian swaps can be directly mapped through the dynamics of a set of zero coupon reference prices that we use in our study. Using these zero coupon yields as observed data, we present clear empirical evidence on the following fact. Suppose we extract the truly stochastic factors (first three state variables) by two distinct methods: First, running sequential linear regressions neglecting the conditionally deterministic factors, and, second, by optimizing the full dynamic model (including the conditionally deterministic factors) by maximum likelihood estimation. Then the two resultant state vectors will be approximations to each other. In addition to this, we also observe and interpret the results for risk premia charged by investors on the Brazilian market.

3.1 The Brazilian Swap Market

One of the most important instruments in the Brazilian fixed income market is the future on DI. In order to understand this contract, we first should define what DI yields are. They are yields obtained from interbank borrowing/lending short term market, daily measured by CETIP (Central de Custodia e de Liquidacao Financeira de Titulos). The DI yield at day t is defined as the average yield obtained in this market in day t, measured as composed yield with daycount basis $\frac{\text{workdays}}{252}$. The future on DI with maturity T is a derivative that at time t costs zero by definition, but whose reference price at time t is obtained by the risk neutral expected value of 100000 Reais discounted by the short term interest rate DI accumulated until maturity T: $P_t = 100000 E_t^*\left(e^{-\int_t^T DI_u du}\right)$, where $E_t^*()$ represents risk neutral expectation conditional on time t, and DI_u denotes the short term interest rate DI at day u. It is very similar to a zero coupon bond, except that being a future, it presents a continuum of cash flows paid at each day, obtained by the difference between the reference price on the day and the reference price on the day before corrected by the DI short term rate of the day before $A_t = P_t - P_{t-1} e^{\int_{t-1}^{t} DI_u du}$. A swap floater-DI versus fixed rate is quoted like a zero coupon bond, in terms of a yield for a fixed maturity, and presents one unique cash flow at its maturity time when the pre-agreed yield (correctly pro-rated) is compared with the short term DI yields cumulated along the life of the swap. At time t, the yield to maturity sw_τ of a swap with maturity at T is obtained by no arbitrage using $P_t = e^{-sw_\tau \tau}$, where $\tau = T - t$ represents the time to maturity of the swap. Bolsa de Mercadorias e Futuros (BM&F) is the entity that offers the futures on DI and swaps contracts, also acting as intermediate for the over-the-counter swaps contracts. It maintains a historical database with interest rates on swaps with different fixed maturities synthetically constructed from the prices of futures on DI with different maturities. The database can be accessed at "www.bmf.com.br".

3.2 Brazilian Swaps Data

Data consists of historical series of Brazilian interest rates swaps for maturities 30, 60, 90, 120, 180, 360 and 720 days, from January 1, 2005 to December 31, 2011. Figure 2 presents the historical evolution of the Brazilian swap data. Almeida (2004) applied Principal Component Analysis (PCA) on the first differences of swap yields and showed that three factors account for 98.7% of the movements of the swap term structure for the period from January 2, 2001 to January 29, 2003. We apply PCA on the larger sample and confirm the fact that three factors explain most of the term structure movements (86.01%). Using this fact, for each day, we apply the static Legendre model of Section 2.1, using the three first Legendre polynomials, constant, linear and quadratic. We give the denomination of Legendre coefficient of degree j to the estimated coefficient multiplying the Legendre polynomial of degree j in the fitting procedure. Figure 3 presents the time series of the three Legendre coefficients. The Legendre coefficient of degree 0 represents the level factor. It has respective mean and standard deviation values of 12.95% and 2.68%. Intuitively, from investors viewpoint, high values of the level factor indicate perception of immediate risk on lending money. The Legendre coefficient of degree 1 represents the slope factor. Its mean and standard deviations are respectively 0.17% and 0.87%. Low values of the slope factor are consistent with flat term structures while high values are consistent with steep term structures and indicate expectation of future risk in lending money for short and medium term maturities. Note on Figure 3 that in 2008 the slope factor achieves its higher value around the Subprime Crisis. The Legendre coefficient of degree 2 represents the torsion factor, and basically indicates the degree of concavity of the term structure. Negative values indicate concavity while positive values convexity. Its mean and standard deviations are respectively 0.04% and 0.32%. Note that along almost the whole sample path the term structure presents convex curvature. The three factors have a high degree of correlation, as usually perception of immediate risk (level factor) comes together with higher expectation of future risks (slope and curvature factors). Table 1 presents the correlation coefficients of the Legendre factors obtained by the Static model.

3.3 Implementing the Gaussian Model

The first important point to be noted is that Y_4, Y_5, and Y_6 are deterministic factors under the Gaussian case. This is a consequence of the combination of two facts: 1) their dynamics do not depend on the Brownian Motion vector, and 2) their drifts do not depend on the first three components of the state vector. Then it is possible that we write down explicitly their dynamics as functions of the time variable, and the model parameters. We adopt the same representation as DS (2002) where the Σ matrix is diagonal in stead of identity[13]. From Equation (28) we directly conclude that:

$$\sigma(Y_t)\sigma(Y_t)^T = L^{-1}\tilde{H}_0(L^{-1})^T = H_0 \tag{36}$$

or equivalently:

$$\tilde{H}_0 = L\Sigma^2((L^{-1})^T)^{-1} = L\Sigma^2 L^T \tag{37}$$

[13]A more general representation for Σ would be an upper triangular matrix, if we assume a diagonal matrix for $\tilde{\Sigma}$. However, using Equation (28) and noting that L^{-1} is upper triangular, we assume $\tilde{\Sigma}$ is also upper triangular restricted in a way that we obtain Σ diagonal.

We use matrix L defined by Equation (49) which uses the fact that the largest maturity of observed Brazilian swaps is 2 years.

Equation (37) combined with Equation (53) inform that the contribution to the drift of each of the state variables Y_4, Y_5, and Y_6 coming from $L^{-1}M$ can be expressed as a linear combination of squared diagonal elements of matrix Σ, as shown in what follows. Explicitly substitute the value of L presented in subsection 2.2:

$$\tilde{H}_0 = \begin{bmatrix} \Sigma_{11}^2 + \Sigma_{22}^2 + \Sigma_{33}^2 & -2\Sigma_{22}^2 - 6\Sigma_{33}^2 & 4.5\Sigma_{33}^2 & 0 & 0 & 0 \\ -2\Sigma_{22}^2 - 6\Sigma_{33}^2 & 4\Sigma_{22}^2 + 36\Sigma_{33}^2 & -27\Sigma_{33}^2 & 0 & 0 & 0 \\ 4.5\Sigma_{33}^2 & -27\Sigma_{33}^2 & 20.25\Sigma_{33}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{38}$$

Use Equations (53) and (38):

$$\begin{aligned} (L^{-1}M)_4 &= 0.2\Sigma_{22}^2 + 2.6751\Sigma_{33}^2 \\ (L^{-1}M)_5 &= 0.2577\Sigma_{33}^2 \\ (L^{-1}M)_6 &= 0.1431\Sigma_{33}^2 \end{aligned} \tag{39}$$

From Equations (27), (52) and (39) obtain the whole drift of the state variables Y_4, Y_5, and Y_6:

$$\begin{aligned} \mu^{\mathbb{Q}}(Y_t)_4 &= 0.2\Sigma_{22}^2 + 2.6751\Sigma_{33}^2 + 8.75Y_{t,5} - 3.15Y_{t,6} \\ \mu^{\mathbb{Q}}(Y_t)_5 &= 0.2577\Sigma_{33}^2 + 10.8Y_{t,6} \\ \mu^{\mathbb{Q}}(Y_t)_6 &= 0.1431\Sigma_{33}^2 \end{aligned} \tag{40}$$

Explicitly solving the simple ODE's implied for these factors, get:

$$Y_{t,4} = Y_{0,4} + (0.2\Sigma_{22}^2 + 2.6751\Sigma_{33}^2 + 8.75Y_{0,5} - 3.15Y_{0,6})t + (1.8041\Sigma_{33}^2 + 94.5Y_{0,6})\frac{t^2}{2} + 13.5231\Sigma_{33}^2\frac{t^3}{6} \tag{41}$$

$$Y_{t,5} = Y_{0,5} + (0.2577\Sigma_{33}^2 + 10.8Y_{0,6})t + 1.5455\Sigma_{33}^2\frac{t^2}{2} \tag{42}$$

$$Y_{t,6} = Y_{0,6} + 0.1431\Sigma_{33}^2 t \tag{43}$$

Note that the dynamics of the state variables $Y_{t,4}$, $Y_{t,5}$ and $Y_{t,6}$ are deterministic and, in addition, completely determined by the parameters Σ_{22} and Σ_{33}, and the initial conditions $Y_{0,4}$, $Y_{0,5}$ and $Y_{0,6}$, which are also treated as parameters of the model.

We use the time series of the Legendre static factors to identify which swaps should be priced without error. Time series of residuals obtained from the static fitting procedure of Section 3.2 indicate that the residuals for maturities 60, 360 and 720 days present the smallest standard deviations. We assume that these swaps yields are priced exactly, and estimate the model by Maximum Likelihood. The swaps for maturities 30, 90, 120 and 180 are priced with i.i.d Gaussian errors (see Dai and Singleton (2002), page 10). The maximum value achieved by the log-likelihood function was 44.41 (for details on ML estimation of a multi-factor Gaussian models see Almeida (2004)). For each time t, the implied stochastic factors, first 3 variables in the state vector $Y_{t,1}, Y_{t,2}, Y_{t,3}$, are extracted using the following linear system:

$$Sw_t^{\text{exact}} - [P_3(x_{\text{exact}})P_4(x_{\text{exact}})P_5(x_{\text{exact}})] \begin{bmatrix} Y_{t,4} \\ Y_{t,5} \\ Y_{t,6} \end{bmatrix} = [P_0(x_{\text{exact}})P_1(x_{\text{exact}})P_2(x_{\text{exact}})] \begin{bmatrix} Y_{t,1} \\ Y_{t,2} \\ Y_{t,3} \end{bmatrix} \tag{44}$$

where Sw_t^{exact} denotes the vector of swap rates priced exactly, $P_i(x_{\text{exact}})$ denotes the Legendre polynomial of degree i evaluated at the vector x_{exact}, and x_{exact} is a vector of transformed maturities $x_{\text{exact}} = \frac{2\tau_{\text{exact}}}{l} - 1$, τ_{exact} being the maturities of the swaps priced exactly.

Figure 4 confirms the fact that the stochastic Legendre factors are not very much affected if obtained by sequentially solving linear regressions using the static model, in stead of by solving the full dynamic model including the deterministic factors. It presents, for the factors attached to the first three Legendre polynomials, level, slope, and curvature, the difference between the dynamic and correspondent static factor. Note that for the three factors the differences through time are all less than 25 bp, with mean and standard deviations of: 1.01 and 4.5 bp for the level factor, -0.53 and 3.05 bp for the slope factor, and, -1.94 and 6.6 bp for the curvature factor.

The deterministic factors given by equations 41, 42 and 43 are shown in Figure 6. The three factors are small: Y_4 varies more, between -13 and 7 bps, with mean of 0.95 bps, Y_5 and Y_6 range between -1 and 3 bps, with mean of 0.2 and -0.03, respectively. In addition, Figure 5 confirms the fact that the deterministic factors have practically no influence on the implied values of the dynamic factors, with differences of less than 1 bp for level and slope factors and differences between -18 and 22 bps for curvature factor, along the whole historical sample path. A plausible explanation for this fact is that the model restricts the initial values $Y_{0,4}$, $Y_{0,5}$ and $Y_{0,6}$, and the parameters $\Sigma_{2,2}$ and $\Sigma_{3,3}$ in a way that the influence of the deterministic factors is minimal. Deterministic factors clearly do not capture well the stochastic behavior of the term structure, only existing for reasons of consistency of the model, imposing some restrictions on the parametric space. However they are important to offer flexibility for the volatility structure of the stochastic factors: Without the deterministic factors the state vector dynamics would be driven by a one-dimensional Brownian Motion. However, as the contribution of the deterministic factors to the term structure movements is very small, we may use the time series obtained through the linear regressions of the static model as an approximation for the factors and then perform a continuous autoregression in the state vector in order to have its dynamics. This is a very interesting approach from the practitioner viewpoint: In addition to providing direct interpretation of the stochastic factors as driving sources for the different types of term structure movements, the model offers an approximation for the factor dynamics without actual implementation of its dynamic version.

Table 2 presents the parameters values, their standard deviations calculated by the Outer Product Method (BHHH), and the ratio $\frac{\text{value}}{\text{std value}}$ which allow the performance of standard asymptotic tests of parameters significance. Bold ratios indicate significant parameters at a 95% confidence level. Table 3 presents statistical properties of the residuals for the maturities 30, 90, 120, 180 and 180 days. The residuals present means and standard deviation values comparable to results presented in the literature using the same estimation method, both for U.S. treasury data (Dai adn Singleton (2000)) and Russian Brady Bonds data (Duffie et al. (2003)).

Figure 7 presents the time series of the sum of squared cross sectional estimated errors (SSE) for the dynamic and static models. The SSE mean is 4.04×10^{-6} and 2.56×10^{-6} for the dynamic and static models, respectively.

The fact that each dynamic factor plays a role as a known movement of the term structure allows a direct interpretation of the risk premia charged by investors in the Brazilian swap market. Whenever analyzing risk premia and changes of measures, for each factor on the model, at least two different effects should be considered: How much the factor dynamics is affected with the change of measure, and how much the factor itself contributes to the prices of risk of each source of uncertainty represented by each entry in the Brownian Motion vector. In this sense, matrix λ_Y indicates that investors perception of risk is related to all factors witch indicates more complex movements of the brazilian term structure.
Figure 8 shows the risk premium time series associated with each factor of the term structure. Time-variation in risk premia is mainly driven by the first and second factors. In other words, risk premia in bond yields mainly depend on the level and slope of the yield curve. All factors have significant effect on the risk premia charged on the first source of uncertainty of the term structure, parameters $\lambda_Y(1,1)$, $\lambda_Y(1,2)$ and $\lambda_Y(1,3)$ are all significant at a 95% confidence. Once $\lambda_Y(1,2)$ is 26 times higher than $\lambda_Y(1,1)$ and the mean absolute level factor is only 15 times higher than the mean absolute slope factor, we can infer that slope is an important component of level risk premia. For instance, during the crisis, when the slope was negative, the level risk premia was higher due to the slope contribution. This is associated with the fact that monetary policy shocks are usually estimated to have a slope effect on the yield curve, affecting short term rates and affecting the risk premia associated with the level. Dewachter and Lyrio (2006) have a nice macroeconomic

interpretation of latent factors in an essentially affine dynamic term structure model. In accordance with their view, we identify the slope as a business cycle factor

Note that although the slope is an important component of level risk-premia it does not significantly affect its own risk premia, that is, the parameter $\lambda_Y(2,2)$ is not significant. Interestingly, when estimating the approximated version of the model (see Section 3.4), we get very similar qualitative and quantitative results for the slope. In particular, the estimated value of $\lambda_Y(2,2)$ based on the approximated method was equal to zero.

Finally, the curvature factor has significant effect on the risk premia charged on all fundamental sources of uncertainty on the term structure[14] directly indicating that this movement plays an important role in the term structure dynamics. This is in line with a PCA analysis performed in the swap dataset which reveals that the curvature PC term accounts for 7% of the movements of this term structure.

For the Gaussian model, the time t instantaneous expected excess return of a swap with maturity τ is given by (see Duffee (2002)):

$$e^i_{t,\tau} = [P_0(\tau)P_1(\tau)P_2(\tau)]\Sigma\Lambda_Y Y_t \tag{45}$$

Equation (45) indicates that the instantaneous expected excess return is a linear combination of some of the factors of the model, where weights on specific factors come from a combination of parameters in matrix λ_Y, Σ and also Legendre polynomial terms which are maturity dependent. However, the most important weights are the ones which come from matrix λ_Y because the only role of these parameters is to capture risk premia, whereas the role of Σ is divided between fitting the cross sectional, minimizing the effect of the deterministic factors, and also capturing the dynamics of yields under P through the transition densities. That is another way of understanding that Λ_Y gives information on which factors are important on the premia charged by investors to hold bond positions.

3.4 Implied Approximated Risk Premia

The Six-Legendre Polynomial Gaussian Dynamic Model presents three risk factors: Level, Slope and Curvature and three deterministic factors, that are completely determined by estimated parameters. The only reason for the existence of the deterministic factors is consistency of the model, they do not capture any feature of the stochastic behavior of the term structure. The estimation results of this model obtained in Section 3.3 showed that the three deterministic factors have practically no influence on the implied values of the

[14] that is, parameters $\lambda_Y(1,3)$, $\lambda_Y(2,3)$ and $\lambda_Y(3,3)$ are all significant at a 95% confidence.

dynamic level, slope, and curvature factors. Once the estimation of the dynamic model isn't very friendly to solve due to optimization issues, we propose an approximation for this model: First we determine the vector of the three risk factors by solving the linear regression as in Equation (3), then we estimate the physical dynamic of this factors by solving a continuous auto regression as in Equation (32). The dynamic under the risk neutral measure $\mathcal{Q}$ is completely determined in Equation (35), once the matrixes L, U, M, defined in Appendix I depends only on features of the model and the physical dynamic parameters. The market prices of risk can then be easily computed by equations (33) and (34).

In order to get comparable results, we solve the first regression only for the yields that are priced without error in the dynamic model. Table 6 shows the mean of the absolute difference between the state vector estimated in the dynamic model and in our approximation: For all three factors we get mean absolute error less than 2 bps, confirming the fact that the deterministic factor has little influence in risk factors estimation. The estimated parameters and the statistical properties of the residuals for the maturities 30, 90, 120 and 180 days are show in Tables 4 and 5, respectively. The estimated parameters are very close to the ones obtained in the full dynamic model estimation as are the residual properties.

3.5 Discussion and Thoughts for Future Research: Separation of the Cross Sectional and Time Series Roles by Approximating the State Vector in More General Specifications of the Model

The Dynamic Legendre Model might be considered an useful model due to the combination of two attractable points: It proposes a parametric form for the term structure resembling the loadings of principal components obtained on the majority of empirical applications of PCA to term structure data; and restricting functions A and B to be polynomials on the time to maturity variable τ, the parameters estimation process becomes focused on the factors dynamics under the physical probability measure P, rather than under Q. The only elements which prevent the estimation process under measures P and Q to be completely separated are the conditionally deterministic factors. This can be noted through Equations (41)-(43) that present the dependence of these conditionally deterministic factors on parameters that define the volatility structure of the whole set of factors. However, in Section 3, we gave empirical evidence for the Gaussian case, that the only role for the conditionally deterministic factors is in fact to create specific restrictions on the vector space in terms of choices for the volatility parameters, not substantially affecting the implied values for the truly stochastic Legendre factors. Based on this observation, we proposed extracting an approximation for the state vector by simply neglecting the conditionally deterministic factors and running cross-sectional regressions with the term structure parameterized by the truly stochastic Legendre polynomials factors.

The above-mentioned approximation allowed us to come up with a simple method to extract interest rate risk premia in a Gaussian polynomial model. However, having the ability to approximate the dynamic state space vector brings a set of other advantages in terms of model specification, even when the model is not necessarily Gaussian:

1) Simple obtention of qualitative information on the dynamics followed by the state vector.

After the extraction of the approximated time series of the state vector, we may adjust $ARMA + GARCH$ models to these time series to test if specifications involving stochastic volatility would be necessary/interesting. We could also use a more sophisticated non-parametric test (Hong and Li (2005)) to test the adequacy of introducing stochastic volatility in the dynamics of the state vector.

2) Simplifying the task of choosing instruments to be priced exactly.

Usually the choice of instruments to be priced exactly is directly related to liquidity in the market. However, when the whole set of observed instruments presents homogeneous liquidity, there is a cost associated to testing all possible combinations of instruments to be priced exactly. When we observe N_{inst} market prices/yields and we wish to fit a N factor model, it gives a total of $C^N_{N_{inst}}$ possibilities (typical values for three factor models will usually range between 20 and 100 combinations). Using the approximated state vector, we propose another practical heuristic methodology to tackle this problem with one simple calculation. The above-mentioned regressions basically choose the coefficients for the Legendre polynomials which minimize the sum of the squared residuals for each fitted yield. In the end, we obtain a time series for each one of the Legendre coefficients and consequently a time series for the residuals of each fitted yield. These residuals are the ones chosen without any dynamic restriction imposed to the model, and only driven by the data available. Now choose the yields which present the smallest residuals standard deviations to be priced exactly. The intuition behind this is that we are allowing the data to guide us on the choice of which elements we should correctly price: precisely the ones which are better priced when we allow all cross sectional yields to be priced with error.

3) Which dynamic factors will be more useful in hedging schemes?

By the fact that each polynomial is related to a type of movement, the variability of each approximated factor from the state vector can be used to identify the order of magnitude of the movements, and to eventually propose hedging schemes without even running the dynamic model. Almeida et al. (2003) showed that the factors will be simple rotations of the significant principal components of the term structure. Then hedging against the movements driven by these polynomial factors is equivalent to hedging against the significant principal components.

4 Conclusion

A parametric arbitrage-free term structure model was implemented, and some empirical results were outlined. The model parameterizes the term structure as a linear combination of Legendre polynomials, which makes it a separable term structure model with the maturity functional dependence being a polynomial function. The form for the factors dynamics presents diffusion terms more general than the standard affine models while drift terms more restricted. We restricted the model to the affine class and showed that even under this particular case, it is represented by a general class of mixed Gaussian+Feller processes, with some conditionally deterministic factors which do not appear in the standard affine class. In particular, in order to test its empirical abilities, we implemented a multi-factor Gaussian version of the model.

The goal was to identify possible advantages of implementing a parametric arbitrage-free model over the standard affine literature. We noted that the previously defined parametric form for the term structure allows us to use simple statistical procedures like running independent daily regressions to extract information regarding the state vector. This information was shown to be useful in identifying the most adequate dynamic model, or in other words, some qualitative characteristics of the term structure dynamics.

Although we only implemented a Gaussian version of the dynamic model, the approximation for the state vector obtained with the cross sectional regressions would also be valid for any other mixed Gaussian+Feller model which could have been implemented. The reason is simple. The difference between the state vector obtained with the regressions and the state vector obtained under any dynamic version of the model is controlled by two elements: The conditionally deterministic factors, and the error terms from the instruments priced exactly under the dynamic model but priced with error under the static (regression) version of the model. The conditionally deterministic factors, on their turn, will always be negligible as long as we choose to implement a model with the number of truly stochastic factors equal to the number of significant principal components of the term structure movements. As for the error terms from the instruments priced exactly under the dynamic model, they will be the same no matter which version of the dynamic model is implemented, because it is the static model that drives them. This turns our independent regression procedure into a robust method for an approximation for the state vector.

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5 Appendix I - Analytical and Numerical Matrices for the Six-Factor Legendre Dynamic Model

This section contains the main matrices that allow for the relation between a power polynomial model and the Legendre polynomial model. For additional details we refer to Almeida and Vicente (2008).

$$L = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & \frac{4}{l} & -\frac{12}{l} & \frac{24}{l} & -\frac{40}{l} & \frac{60}{l} \\ 0 & 0 & \frac{18}{l^2} & -\frac{90}{l^2} & \frac{270}{l^2} & -\frac{630}{l^2} \\ 0 & 0 & 0 & \frac{80}{l^3} & -\frac{560}{l^3} & \frac{2450}{l^3} \\ 0 & 0 & 0 & 0 & \frac{350}{l^4} & -\frac{3150}{l^4} \\ 0 & 0 & 0 & 0 & 0 & \frac{1512}{l^5} \end{bmatrix} \tag{46}$$

$$M = \begin{bmatrix} 0 \\ \tilde{H}_{0,11} \\ \frac{\tilde{H}_{0,12}}{2} + \tilde{H}_{0,21} \\ \frac{\tilde{H}_{0,22}}{2} + \tilde{H}_{0,31} \\ \frac{\tilde{H}_{0,23}}{3} + \frac{\tilde{H}_{0,32}}{2} \\ \frac{\tilde{H}_{0,33}}{3} \end{bmatrix} \tag{47}$$

$$U_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{48}$$

$$L = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 2 & -6 & 12 & -20 & 30 \\ 0 & 0 & 4.50 & -22.50 & 67.50 & -157.50 \\ 0 & 0 & 0 & 10 & -70 & 306.50 \\ 0 & 0 & 0 & 0 & 21.88 & -196.88 \\ 0 & 0 & 0 & 0 & 0 & 47.25 \end{bmatrix} \tag{49}$$

$$L^{-1} = \begin{bmatrix} 1 & 0.5000 & 0.4444 & 0.5000 & 0.6400 & 0.6111 \\ 0 & 0.5000 & 0.6667 & 0.9000 & 1.2800 & 1.4048 \\ 0 & 0 & 0.2222 & 0.5000 & 0.9143 & 1.3095 \\ 0 & 0 & 0 & 0.1000 & 0.3200 & 0.6852 \\ 0 & 0 & 0 & 0 & 0.0457 & 0.1905 \\ 0 & 0 & 0 & 0 & 0 & 0.0212 \end{bmatrix} \tag{50}$$

$$L^{-1}M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \tag{51}$$

$$L^{-1}UL = \begin{bmatrix} 0 & 2 & -1.5000 & 2.8333 & -2.0833 & 38.2833 \\ 0 & 0 & 4.5000 & -2.5000 & 6.2500 & 48.6500 \\ 0 & 0 & 0 & 6.6667 & -2.9167 & 26.4167 \\ 0 & 0 & 0 & 0 & 8.7500 & -3.1500 \\ 0 & 0 & 0 & 0 & 0 & 10.8000 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{52}$$

$$L^{-1}M = \left[\begin{array}{l} 0.5\tilde{H}_{0,11} + 0.4444\left(\frac{\tilde{H}_{0,12}}{2} + \tilde{H}_{0,21}\right) + 0.5\left(\frac{\tilde{H}_{0,22}}{2} + \tilde{H}_{0,31}\right) + 0.64\left(\frac{\tilde{H}_{0,23}}{3} + \frac{\tilde{H}_{0,32}}{2}\right) + 0.6111\left(\frac{\tilde{H}_{0,33}}{3}\right) \\ 0.5\tilde{H}_{0,11} + 0.6667\left(\frac{\tilde{H}_{0,12}}{2} + \tilde{H}_{0,21}\right) + 0.9\left(\frac{\tilde{H}_{0,22}}{2} + \tilde{H}_{0,31}\right) + 1.28\left(\frac{\tilde{H}_{0,23}}{3} + \frac{\tilde{H}_{0,32}}{2}\right) + 1.4048\left(\frac{\tilde{H}_{0,33}}{3}\right) \\ 0.2222\left(\frac{\tilde{H}_{0,12}}{2} + \tilde{H}_{0,21}\right) + 0.5\left(\frac{\tilde{H}_{0,22}}{2} + \tilde{H}_{0,31}\right) + 0.9143\left(\frac{\tilde{H}_{0,23}}{3} + \frac{\tilde{H}_{0,32}}{2}\right) + 1.3095\left(\frac{\tilde{H}_{0,33}}{3}\right) \\ 0.1\left(\frac{\tilde{H}_{0,22}}{2} + \tilde{H}_{0,31}\right) + 0.32\left(\frac{\tilde{H}_{0,23}}{3} + \frac{\tilde{H}_{0,32}}{2}\right) + 0.6852\left(\frac{\tilde{H}_{0,33}}{3}\right) \\ 0.0457\left(\frac{\tilde{H}_{0,23}}{3} + \frac{\tilde{H}_{0,32}}{2}\right) + 0.1905\left(\frac{\tilde{H}_{0,33}}{3}\right) \\ 0.0212\left(\frac{\tilde{H}_{0,33}}{3}\right) \end{array} \right. \tag{53}$$

Factor	Level	Slope
Level	1.00	
Slope	-0.54	1.00
Curvature	-0.13	-0.59

Table 1: Correlation Structure of the Legendre Static Factors.

Parameter	Value	Standard Error	ratio $\frac{abs(\text{Value})}{\text{Std Err.}}$
κ_{11}	0.141	0.046	**3.05**
κ_{12}	-1.608	1.468	**1.10**
κ_{22}	0.557	1.355	0.41
κ_{13}	10.303	3.781	**2.72**
κ_{23}	-4.710	3.274	1.44
κ_{33}	3.247	1.340	**2.42**
Σ_{11}	0.014	0.0001	**107.15**
Σ_{22}	0.012	0.0002	**77.30**
Σ_{33}	0.009	0.0000	**214.68**
$\lambda_0(1)$	0.005	0.564	0.01
$\lambda_0(2)$	0.018	1.063	0.02
$\lambda_0(3)$	0.016	0.356	0.05
$\lambda_Y(1,1)$	10.30	5.261	**1.96**
$\lambda_Y(1,2)$	-263.94	106.700	**2.47**
$\lambda_Y(2,2)$	47.07	113.755	0.41
$\lambda_Y(1,3)$	863.39	263.647	**3.27**
$\lambda_Y(2,3)$	-778.77	276.014	**2.82**
$\lambda_Y(3,3)$	374.43	118.159	**3.17**
$Y_{0,4}$	-0.0014	0.0001	**10.62**
$Y_{0,5}$	0.0003	0.00001	**20.74**
$Y_{0,6}$	-0.00003	0.000001	**51.93**

Table 2: Parameters and Standard Errors for the Gaussian Model Using Brazilian Data.

Residuals	Mean (bp)	Mean Abs (bp)	Standard Deviation (bp)	Skewness	Kurtosis
$\hat{\epsilon}_{30}$	1.75	6.80	8.87	0.36	5.05
$\hat{\epsilon}_{90}$	-1.89	5.49	6.81	0.37	4.60
$\hat{\epsilon}_{120}$	-1.04	8.54	10.68	0.16	3.78
$\hat{\epsilon}_{180}$	-3.83	10.25	12.12	0.53	3.30

Table 3: Statistical Properties of the Residuals of the Gaussian Model.

Parameter	Value	Standard Error	ratio $\frac{abs(\text{Value})}{\text{Std Err.}}$
κ_{11}	0.118	0.046	**3.05**
κ_{12}	-1.662	1.468	**1.10**
κ_{22}	0.000	1.355	0.41
κ_{13}	9.270	3.781	**2.72**
κ_{23}	-4.710	3.274	1.44
κ_{33}	3.777	1.340	**2.42**
Σ_{11}	0.014	0.0001	**107.15**
Σ_{22}	0.012	0.0002	**77.30**
Σ_{33}	0.009	0.0000	**214.68**
$\lambda_0(1)$	0.007	0.564	0.01
$\lambda_0(2)$	0.021	1.063	0.02
$\lambda_0(3)$	0.018	0.356	0.05
$\lambda_Y(1,1)$	8.632	5.261	**1.96**
$\lambda_Y(1,2)$	-267.95	106.700	**2.47**
$\lambda_Y(2,2)$	0.000	113.755	0.41
$\lambda_Y(1,3)$	788.06	263.647	**3.27**
$\lambda_Y(2,3)$	-777.94	276.014	**2.82**
$\lambda_Y(3,3)$	414.25	118.159	**3.17**

Table 4: Parameters and Standard Errors for the Approximated Gaussian Model Using Brazilian Data.

Residuals	Mean (bp)	Mean Abs (bp)	Standard Deviation (bp)	Skewness	Kurtosis
$\hat{\epsilon}_{30}$	2.01	7.06	9.07	0.16	4.81
$\hat{\epsilon}_{90}$	-1.66	5.14	6.54	0.17	4.70
$\hat{\epsilon}_{120}$	-0.88	8.34	10.48	0.11	3.84
$\hat{\epsilon}_{180}$	-3.84	10.33	12.20	0.56	3.35

Table 5: Statistical Properties of the Residuals of the Approximated Gaussian Model.

Factors	Mean Abs (bp)
Level	0.463
Slope	1.899
Curvature	0.934

Table 6: Mean Absolute Difference of the State Factors in The Approximated Gaussian Model.

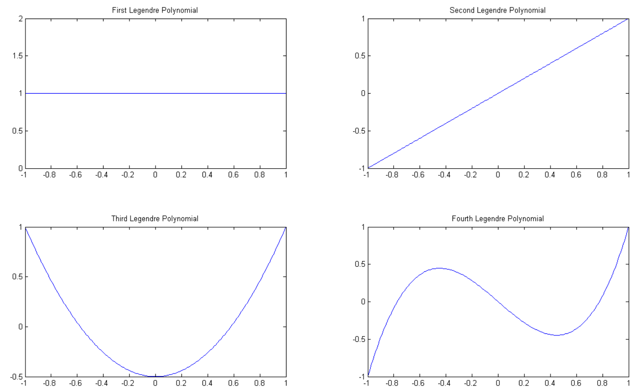

Figure 1: Legendre Polynomials.

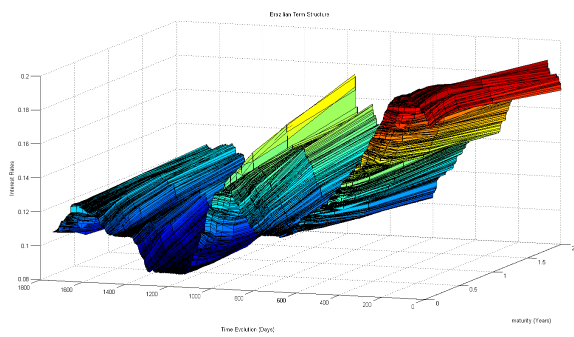

Figure 2: Temporal Evolution of the Brazilian Term Structure.

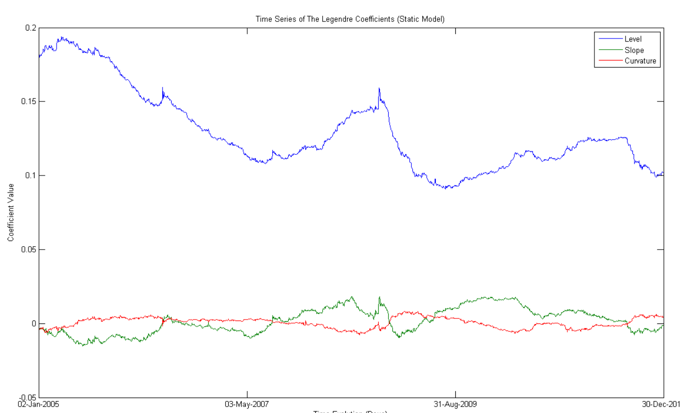

Figure 3: Static Model: Temporal Evolution of the Legendre Coefficients for the Brazilian Swaps.

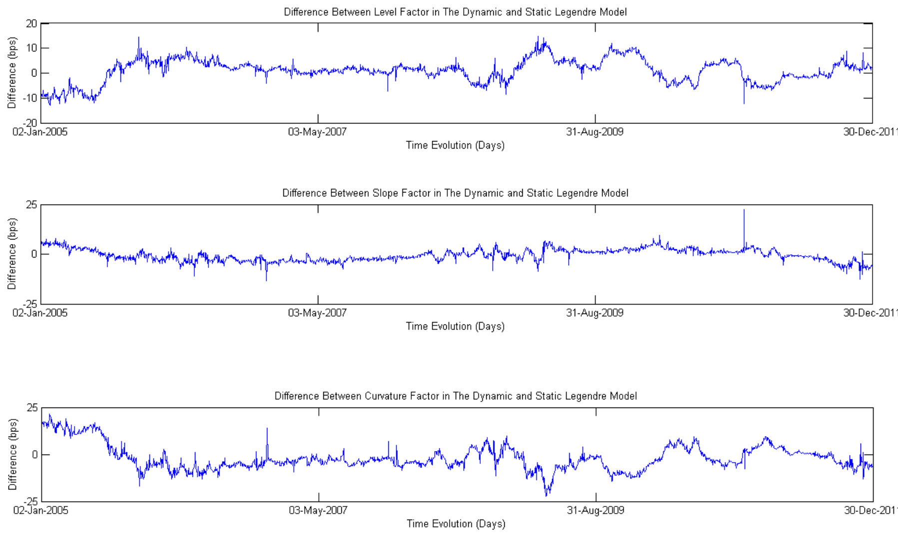

Figure 4: Difference Between Legendre Dynamic and Static Factors for the Gaussian Model.

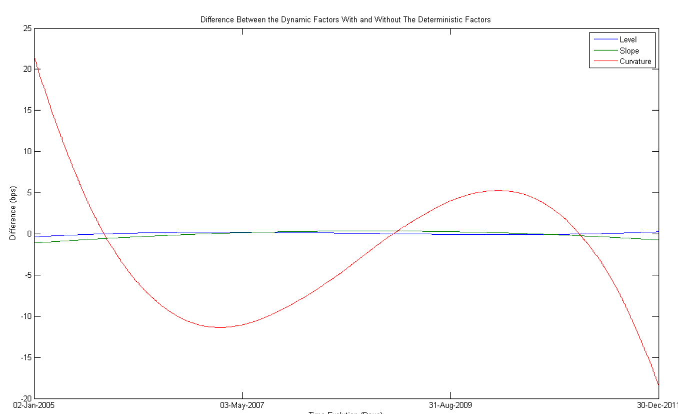

Figure 5: Effect of the Conditionally Deterministic Factors on the Values of the Stochastic Factors on the Gaussian Model.

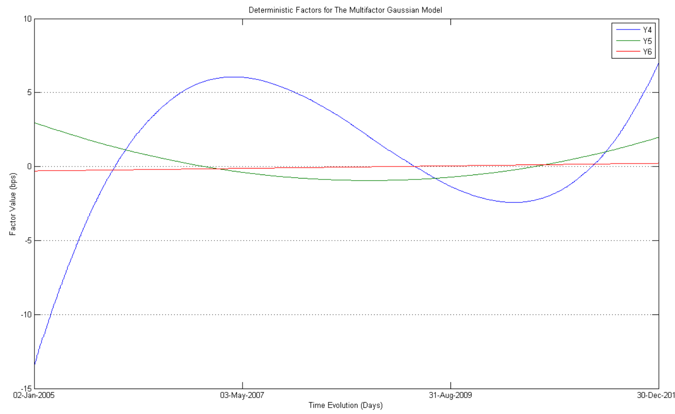

Figure 6: Conditionally Deterministic Factors in the Multi-factor Gaussian Model.

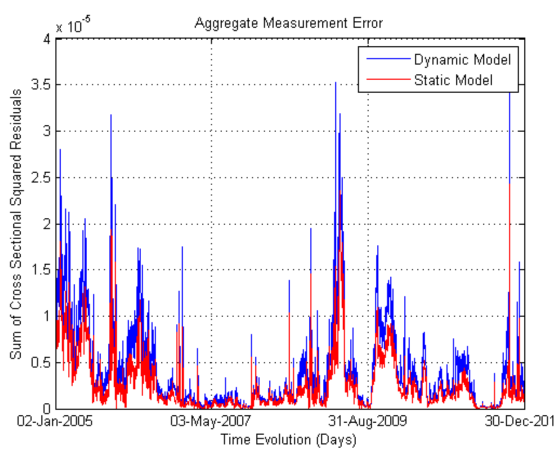

Figure 7: Time Series of the Aggregate Measurement Error for the Gaussian Model and Static Model.

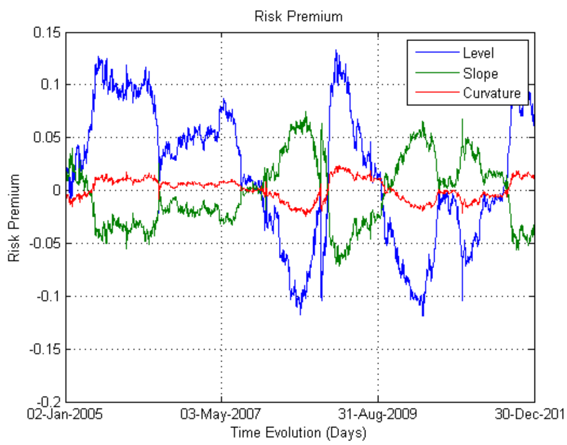

Figure 8: Time Series of the Risk Premium Term for Each Factor.