Pricing interest rate derivatives under monetary changes

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Abstract

Traders worldwide use interest rate options and futures to bet on future monetary decisions, in particular in countries where the monetary regime is Inflation Targeting (IT). Under an IT regime Central Banks tend to define the target rate on scheduled meetings. We propose in this paper a simple and consistent way to explicitly incorporate the potential changes in the target rate during Central Bank's meetings into interest rate futures and option pricing. We calibrate the model to data from Brazil where there is a liquid market for futures and options on overnight interest rate.

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1 Introduction

According to BIS semiannual OTC derivatives statistics report released on June 2012 the interest rate derivatives market represents 77% of all notional amounts outstanding worldwide, BIS (2012). Many participants in this market are hedging their positions that have an interest rate risk with an offsetting derivative contract. On the other hand, another group of participants will use interest rate derivatives to take risk. For instance, interest

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rate options products provide market participants the right payoff to bet on Central bank futures decisions about the target rate, namely in countries under a Inflation Targeting (IT) regime. For instance, if a binary options is available, investors can make bets on futures values of the overnight rate at time t by buying/selling binary options expiring in the following business day after a scheduled meeting u.

Recently, in response to the ongoing Libor scandal, which has revealed that this important global benchmark of interest rates was manipulated by parties tasked with setting the rate, we observe a significant part of the interest rate market migrating their interest rate exposures to overnight interest rate derivatives, in particular overnight indexed swaps, OIS. An overnight indexed swap (OIS) is an interest rate swap where the floating rate of the swap is equal to the geometric average of an overnight index rate over every day of the payment period. The index rate is typically a central bank rate or equivalent, for example the Federal funds rate in the US.

The empirical literature about the predictability of monetary changes using derivatives is vast. Ederington and Lee (1996) analyze the response of options on Treasury, Eurodollar, and foreign exchange futures to a number of different macroeconomic announcements using an approach similar to Patell and Wolfson (1979, 1981). They find that implied volatility increases on days without announcements and decreases after a wide range of macroeconomic announcements. Beber and Brandt (2004) find that the risk-neutral skewness and kurtosis embedded in Treasury bond futures options change around scheduled macroeconomic announcements, in addition to documenting that implied volatility decreases after the announcements. There are also a number of papers that analyze the impact of scheduled announcements on equity options. Dubinsky and Johannes (2004) extract estimates of the uncertainty embedded in earnings announcements using option prices. They reduce the pricing errors by developing a no-arbitrage option pricing model incorporating deterministic timed jump occurring at the earnings release.

Our paper is closely related, at least on an intuitive level, to Piazzesi (2005) where the author describes the Feds target as a pure jump process and jump intensities depend on the state of the economy and the meeting calendar of the Federal Open Market Committee (FOMC).

On the theoretical side, the goal of this paper is to develop a tractable reduced form model incorporating jumps on Central Bank meetings to price derivatives on overnight interest rate. The key element in our model, unlike traditional interest rate models, is the fact that we disentangle the overnight rate into two components, the first is a continuous processes governing the overnight rate between two scheduled meeting and a second one formed by a deterministically timed jump describing Central Bank meetings outcome.

The rest of this paper is organized as follows. Section 2 presents the paper motivation's, Section 3 describes how to model forthcoming monetary decisions using a discrete time Markov Chain. Section 4 presents closed formula solutions for pricing interest rate futures and options incorporating the market expectations about future changes in the monetary policy. Section 5 describes the model's calibration to Brazilian data and Section 6 concludes.

2 Motivation

In the USA and many other jurisdictions meeting days for the Monetary Authority, FOMC meeting for instance, are marked as special events on the calendars of many market participants because changes in the target rate tend to impact investments' profits. Central Bank meetings are considered special days to market participants worldwide, in countries like Brazil, Australia and England, which have adopted inflation target (IT) regime to conduct the monetary policy, market participants track closely these scheduled events. To properly incorporate the this feature we start by assuming an arbitrary process to describe the dynamic of overnight (spot) interest rate, $(R_t)_{t\geq 0}$. It is a well know result that under no-arbitrage the zero coupon bond (ZCB) price at time t and expiration at T is given by, $P(t,T)$:

$$P(t,T) = \mathbb{E}(e^{-\int_t^T R_s ds}|\mathcal{G}_t) \tag{1}$$

Under a standard continuous affine framework, zero coupon bond prices can be obtained using the conditional Characteristic function (ChF), as Duffie et al. (2003). On the other hand, if there is a scheduled Central Bank meeting before the bond maturity, interest rate must reflect this, otherwise the bond will be mispriced. A feasible way to incorporate Central Bank's decisions regarding the target rate is by assuming that the resulting overnight rate is a semimartigale where the discontinuous component captures monetary decisions. However, semimartigale assume jump occurrence are doubly stochastic, in a sense that both dimension for the point process, i.e. the jump time and the jump magnitude, are stochastic. However, randomly timed jump is an assumption not consistent for modeling interest rate products in a presence of scheduled meeting by the Monetary authority.

Before formally construct our model we offer some insights by assuming that one knows exactly the jump size on a scheduled meeting ($t = u$), therefore we can rewrite equation (1) as :

$$P(t,T) = \mathbb{E}(e^{-\int_t^T r_s ds}|\mathcal{G}_t)e^{-\theta(T-u)} \tag{2}$$

where θ is the monetary decision on time u.

Additionally, if one believe that the overnight interest rate is kept fixed between two consecutive scheduled meetings we obtain:

$$P(t,T) = e^{-r(T-t)}e^{-\theta(T-u)} \tag{3}$$

Even though equation (3) relies on an unrealistic assumption of no uncertainty on the economy its multiplicative form will be also found when interest rate and monetary decisions are stochastic.

Generically, we can set that the overnight interest rate dynamics is given by:

$$\mathrm{d}r_t = \mu(r_t)\mathrm{d}t + \sigma(r_t)\mathrm{d}W_t, \quad t > 0. \tag{4}$$

Where $\mu(\cdot)$ is the drift and $\sigma(\cdot)$ is the diffusion coefficient.

To incorporate scheduled events we assume that interest rate processes have a deterministically timed jump occurring at Central Bank meetings. Thus we assume that θ_u describes a stochastic process which reflects changes in the target rate defined by the Central Bank and released at day u, where $t \leq u \leq T$. In practice we observe that θ_t tend to assume values multiples of some known quantity, for instance 25 basis-point (i.e. 0.25%).

So we set that the observable interest rate process, R_t, can be decomposed into:

$$R_t = r_{t^-} + \theta_t \tag{5}$$

where $r_{t^-} = \lim_{s \uparrow t} r_t$ describes the overnight interest rate immediately before time t and θ_t the outcome from Central Bank. Therefore between two scheduled meetings, the overnight rate evolves as a diffusion process and jumps by a random size θ at scheduled meeting.

At first sight, the expression (5) might seem too restrictive, however it is important to keep in mind that overnight interest rate are determined by interbank transactions and there is no reason to believe that, without any deterioration in commercial banks' credit quality, the new target rate will change the dynamics of the borrow/lending rate among banks with same creditworthiness. So, once the Central bank release the value θ_u at u, the overnight rate jumps to the new level and afterward fluctuate in a diffuse way.

To empirically support our model, the figure below exhibits the recent evolution of the overnight interest rate in Brazil, which, as mentioned before, has an Inflation Target regime since 1999 and all[1] changes on target rate are defined on scheduled meeting.

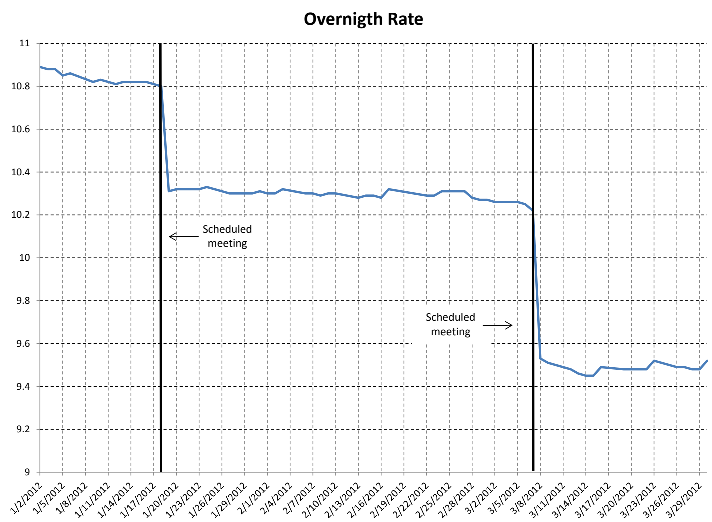

Figure 1: Overnight interest rate evolution in Brazil

For the selected scheduled meeting of the Brazilian Central Bank, we can observe the pronounced effect of jumps on overnight rate. This findings supports our assumption that the observable interest rate can be decomposed into two components, a continuous process describing the overnight rate between meetings and a point process which captures the monetary

[1]Technically speaking the Central Bank can call an extraordinary meeting anytime. However since 1999, the central bank modified the target rate in an extraordinary meeting just once in a total of 117 regular meetings.

decisions.

Based on our assumptions, we investigate first the conditional characteristic function (ChF) of spot rate R_t:

$$\begin{aligned}\phi(u,R_t,t,T) =&\mathbb{E}\left(e^{-\int_t^T R_s \mathrm{d}s + iuR_T}|\mathcal{G}_t\right) &(6)\\ =&\mathbb{E}\left(e^{-\int_t^T \theta_s \mathrm{d}s + iu\psi_T}\cdot e^{-\int_t^T r_s \mathrm{d}s + iur_T}|\mathcal{G}_t\right) &(7)\end{aligned}$$

Additionally, if processes θ_t and r_t are independent we can obtain:

$$\phi(u,R_t,t,T) = \phi(u,\theta_t,t,T)\cdot\phi(u,r_t,t,T) \tag{8}$$

According to Duffie et ali.(2000) the discounted ChF, $\phi(u,r_t,\tau) := \phi(u,r_t,t,T)$ for an affine interest rate model for $u\in\mathbb{C}$ is:

$$\phi(u,r_t,\tau) = e^{A(u,\tau)+B(u,\tau)r_t} \tag{9}$$

with $\tau = T-t$ and initial condition $A(u,0) = 0$ and $B(u,0) = iu$.

If one knows the evolution of $(\theta_t)_{t\geq 0}$ it becomes a deterministic time dependent function and a zero-coupon bond at time t expiring at T, $P(t,T)$, can be written as a product of a deterministic factor and the bond price in an ordinary affine model with $u = 0$ in equation (6):

$$P(t,T) = \exp\left(-\int_t^T \theta_s \mathrm{d}s + A(0,\tau) + B(0,\tau)r_t\right) \tag{10}$$

The component $\int_t^T \theta_s \mathrm{d}s$ captures all possible scheduled meeting outcomes over the interval $[t,T]$. We also observe that we meet the boundary condition $P(T,T) = 1$. Even though equation (10) relies on an unrealistic assumption of no uncertainty on monetary outcomes its form will be also found when interest rate and monetary decisions are stochastic.

A key element for describing the monetary policy, in particular for countries under IT-regime, is that values of $(\theta_t)_{t\geq 0}$ are not independent through the time but its values tend to reflect the current monetary policy pursued by the Central Bank. Under this hypothesis the Central bank tend to increase or decrease the target rate by multiples of 25 Bps, or even keep it unchanged, so $\theta = 0$. Figure 2 exhibits the time dependence on Monetary decisions between two consecutive meeting for the Brazilian Central Bank.

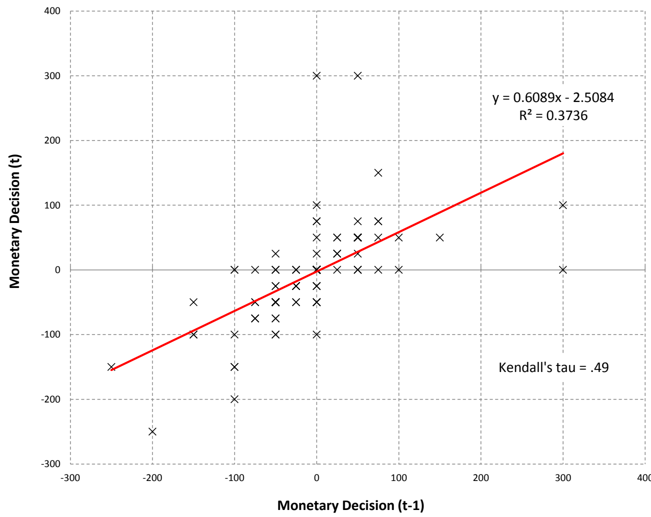

Figure 2: Persistence on Monetary decision in Brazil - Sample from 2000 to 2012

Figure 2 depicts the dispersion diagram between two consecutive meeting. Although we present the linear fit for the data, the better measure to describe persistence in this context is the Kendall's tau which expresses the similarity of the orderings of the data when ranked by each of the quantities. The Kendall's tau between two consecutive meeting in our sample is close to 0.5 which represents an indicative that these processes exhibits some level of persistence. The dependence found suggests that we should model the evolution of the Monetary decisions by Central Banks, $(\theta_t)_{t\geq 0}$. A feasible way to incorporate simultaneously uncertainty and dependence on central bank decisions is by employing a Discrete Time Markov Chain (DTMC) of order k for modeling $(\theta_t)_{t\geq 0}$.

A second potential factor impacting $(\theta_t)_{t\geq 0}$ could be the current overnight interest rate $(r_t)_{t\geq 0}$. Recurring again to Brazilian data, we exhibit the relation between the central bank decision on times u and the current overnight interest rate:

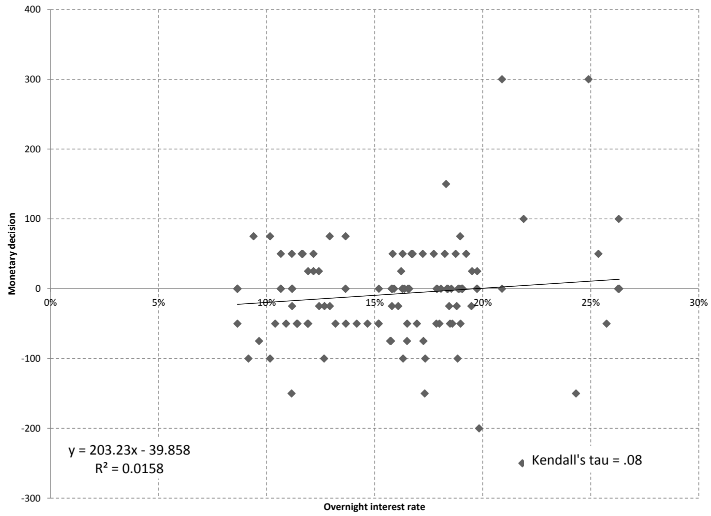

Figure 3: Target and Overnight interest rate evolution - Sample from 2000 to 2012

Differently from figure 2 we do not find, either visual or using Kendall's tau, any evidence that monetary decision depends on the current overnight rate. However from this analysis, we do have some elements to state that the process $(\theta_t)_{t\geq 0}$ itself depends on the monetary cycle pursued by the Central Bank and therefore is not temporally independent. In another words, in a loose (tight) monetary cycle the probability of observing two reductions (increases) in a row is higher that two consecutive decision with opposite signs.

Additionally, at time t, θ_u is not adapted to $\mathcal{G}_t$ so market participants need to estimate θ_u to price an interest rate linked instrument. In next sections, we impose some structure on $(\theta_t)_{t\geq 0}$ and its usage to price overnight interest rate instruments in a closed-form.

3 Modeling $(\theta_t)_{t\geq 0}$ as a DTMC

For sake of simplicity we assume that $(\theta_t)_{t\geq 0}$ is an ergodic Markov Chain of order one. A Markov chain is called ergodic if there exists t such that for all $x, y \in \Omega$, $\mathbb{P}^t(x, y) > 0$. For finite Markov chains the following pair of conditions are equivalent to ergodicity:

1. Irreducible: For all $x, y \in \Omega$, there exists t = t(x, y) such that $\mathbb{P}^t(x, y) > 0$;
2. Aperiodic: For all $x \in \Omega$, $\gcd\{t : \mathbb{P}^t(x, x) > 0\} = 1$.

These assumption are not too restrictive because: first, one can always write a k order DTMC as a first order DTMC, second periodicity is not a rational behavior under a IT-regime and third the set $\mathcal{A}$ given by all potential values of Central Bank's decision about $(\theta_t)_{t\geq 0}$ is finite.

Usually θ_t tend to assume values multiples of some known quantity, for instance 25 Basis-point (Bps). Therefore we define $\mathcal{A}$ as the set of possible outcomes in one Central Bank meeting. Typical elements of $\mathcal{A}$ are $i = k \times 0.0025$ such that $k \in \mathbb{Z}$. Additionally, once θ is DTMC its marginal distribution $\mathbb{P}(\theta_u = i)$ over $\mathcal{A}$ at time u is described by[2]:

$$\mathbb{P}(\theta_u = i) = \sum_j \mathbb{P}(\theta_u = i|\theta_s = j)\mathbb{P}(\theta_s = j) \tag{11}$$

Where transition probabilities $\mathbb{P}(\theta_u = i|\theta_t = j)$ satisfy the Chapman Kolmogorov equation for two consecutive Central Bank meetings $s < t < u$.

A convenient simplification arise in equation (11) when there exist just one scheduled meeting before the bond maturity. In this case, $\theta_s \in \mathcal{G}_t$ and equation (11) simplifies to:

$$\mathbb{P}(\theta_u = i) = \mathbb{P}(\theta_u = i|\theta_s = j) \tag{12}$$

Because $\mathbb{P}(\theta_s = j)$ assume just two outcomes $\{0, 1\}$. We have $\mathbb{P}(\theta_s = j) = 1$ if $\theta_s = j$ was the decision taken by Central bank at meeting s and zero otherwise. Such simplification is important to calibrate the transition probabilities from market prices.

[2]A technical question could arises when dealing with DTMC evolution. Equation (**??**) describes the probability for the process be at state $\theta_u = 1$ after n steps, for our purpose we might need the probability that the process hit by the first time the state $\theta_u = 1$. Even though conceptually different this distinction is not relevant when dealing with DTMC that walks few steps as in our case.

4 Pricing interest rate instruments

Due to their importance to the interest rate market we present in this section closed form solution for zero coupon-bonds and overnight interest rate options. To reach our goal we set the continuous overnight rate $(r_t)_{t\geq 0}$ as follows:

$$\mathrm{d}r_t = \kappa(\Theta - r_t)\mathrm{d}t + \sigma \mathrm{d}W_t \tag{13}$$

Equation (13) is the standard mean reversion Gaussian interest rate model developed initially by Vasicek (1978). According to our model's assumptions, the observable overnight interest rate is the result of two components, the first one is a continuous overnight rate r_t process which describes the overnight rate evolution between two central bank meeting and the second one captures monetary decisions, θ_t:

$$R_t = r_{t^-} + \theta_t \tag{14}$$

Where the evolution of $(r_t)_{t\geq 0}$ is given by (13) while $(\theta_t)_{t\geq 0}$ evolve as a Discrete Time Markov Chain as defined at section 3.

An intermediate result relevant for pricing overnight interest rate products is the next lemma:

Lemma 1 *If* $\mathrm{d}r_t$ *is a Vasicek process then:*

$$-\int_t^T r_s \mathrm{d}s \sim Normal(M(t,T), V(t,T) \tag{15}$$

where:

$$M(t,T) = \frac{r_t - \Theta}{\kappa}\left(1 - e^{-\kappa\tau}\right) - \kappa\tau \tag{16}$$

$$V(t,T) = \frac{\sigma^2}{2\kappa^3}\left(2\kappa\tau - 3 + 4e^{-\kappa\tau} - e^{-2\kappa\tau}\right) \tag{17}$$

We do not prove this lemma because its proof is well known[3].

[3] However the interested reader can consult Mamon (2004) for instance.

4.1 Zero-coupon bond pricing

According to our assumptions, the price at time t of a zero-coupon bond maturing at time T is:

$$
\begin{aligned}
P(t,T) &= \mathbb{E}(e^{-\int_t^T R_s \mathrm{d}s} | \mathcal{G}_t) &(18)\\
&= \mathbb{E}(e^{-(\int_t^T r_s \mathrm{d}s + \int_t^T \theta_s \mathrm{d}s)} | \mathcal{G}_t) &(19)
\end{aligned}
$$

Where $t < u < T$ and u is the scheduled meeting.

To calculate the zero-coupon price above we need to solve the expectation over two stochastic process $(r_t)_{t\geq 0}$ and $(\theta_t)_{t\geq 0}$. In section 2 we assumed that the evolution of $(\theta_t)_{t\geq 0}$ were known in advance and the ZCB price was obtained as:

$$
P(t,T) = \exp\left(-\int_t^T \theta_s \mathrm{d}s + A(0,\tau) + B(0,\tau) r_t\right) \tag{20}
$$

However, in practice this quantity is random and the expectation in (18) is calculated over the joint density of (r_T, θ_T) which might be quite complicate because θ_T is a DTMC and therefore the joint density will be a mixture of continuous and discrete variables. So we state:

Proposition 1 *The no-arbitrage price of a zero-coupon bond is given by:*

$$
P(t,T) = \sum_i \exp\left(-\int_t^T \theta_s \mathrm{d}s + A(0,\tau) + B(0,\tau) r_t\right) \mathbb{P}(\theta_T = i) \tag{21}
$$

where $A(0,\tau)$ and $B(0,\tau)$ are standard Vasicek coefficients given by:

$$
\begin{aligned}
B(0,\tau) &= -\frac{1 - e^{-\kappa\tau}}{\kappa} &(22)\\
A(0,\tau) &= \left(\Theta - \frac{\sigma^2}{2\kappa^2}\right)[B(0,\tau) - \tau] - \frac{\sigma^2 B(0,\tau)^2}{4\kappa} &(23)
\end{aligned}
$$

and $\mathbb{P}(\theta_{T,i})$ are calculated by (11).

Proof of Proposition 1. We start our proof by rewriting equation (19) as:

$$P(t,T) = \int_{\mathcal{A}\times\Omega} [(e^{-(\int_t^T r_s \mathrm{d}s + \int_t^T \theta_s \mathrm{d}s)}] \mathrm{d}F(r_i, \theta) \tag{24}$$

$$\int_{\mathcal{A}} \left[\int_{\Omega} [e^{-(\int_t^T r_s \mathrm{d}s + \int_t^T \theta_s \mathrm{d}s)}] \mathrm{d}F(r_i|\theta) \right] \mathrm{d}G(\theta) \tag{25}$$

where for a fixed $\int_t^T \theta_s \mathrm{d}s$ we can write:

$$P(t,T) = \int_{\mathcal{A}} \left[e^{\int_t^T \theta_s \mathrm{d}s} \int_{\Omega} [e^{-(\int_t^T r_s \mathrm{d}s)}] \mathrm{d}F(r_i|\theta) \right] \mathrm{d}G(\theta) \tag{26}$$

where the inner integral is calculated using the conditional Characteristic function (ChF) $\phi(0, r_t, \tau)$ which for the Vasicek model provides closed form solution to $A(0,\tau)$, $B(0,\tau)$ as described above. So for a given θ_T we can solve (19) as a classical ZCB pricing in a Vasicek model with a deterministic time-dependent drift. Finally, the ZCB price is obtained by calculating the outer integral, which consists in repeating the first step over all possible values of θ_T weighted by its probability.

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4.2 Options Pricing

Binary options provide market participants the right payoff to bet on Central bank futures decisions about the target rate. Binary options pays out one unit of cash if the overnight interest rate R_t is equal or above the strike at maturity. Binary options are generally considered "exotic" instruments and there is no liquid market for trading these instruments between their issuance and expiration. The lack of liquidity to unwind a position before the maturity make binary options less appealing in practice, because sometimes traders may need readjust their position after a new economic indicator, which may impact Central Bank decision on $(\theta_t)_{t\geq 0}$, is released.

Exchanged-traded interest rate options tend to be plain vanilla, for instance CME Group has both futures and options on 30-Day Fed Funds. The contracts are designed to speculate/hedge on changes in short-term interest rates brought about by changes in Federal Reserve monetary policy. As observed earlier, part of the USA interest market has switched to overnight interest rate derivatives, such as overnight indexed swaps, OIS. An overnight indexed swap (OIS) is an interest rate swap where the periodic floating rate of the swap is equal to the geometric average of an overnight index rate over every day of the payment period. Besides swaps other derivatives can have overnight rates as underlying, options for instance. In fact, we can point out IDI options traded at Brazilian Securities and Futures Exchange, BM&FBOVEPSA, as an example of overnight indexed option.

The underlying asset for IDI options is the IDI index defined as the accumulated overnight interest rate $(R_t)_{t\geq 0}$. Therefore, if we associate the continuously-compounded overnight interest rate to $(R_t)_{t\geq 0}$, then IDI is given by:

$$\mathrm{IDI}_T = \mathrm{IDI}_t e^{\int_t^T R_s \mathrm{d}s} \tag{27}$$

An IDI option with maturity T is an European option whose payoff depends on the integral of the overnight rate through time t and option expiration date T.

Denote by $Call(T, K, R_t)$ the time t price of a call option on the IDI, with maturity T and strike price K. Then:

$$Call(T, K, R_t) = \mathbb{E}\left[e^{-\int_t^T R_s \mathrm{d}s}(IDI_T - K)^+ | \mathcal{F}_t\right] \tag{28}$$

Expression (28) can be simplified after plugging (27) and (5):

$$\begin{aligned}
Call(T, K, R_t) &= \mathbb{E}\left[e^{-\int_t^T R_s \mathrm{d}s}(IDI_T - K)^+ | \mathcal{F}_t\right] && (29)\\
&= \mathbb{E}\left[e^{-\int_t^T (r_s \mathrm{d}s + \theta_s \mathrm{d}s)}\left(IDI_t e^{\int_t^T (r_s \mathrm{d}s + \theta_s \mathrm{d}s)} - K\right)^+ | \mathcal{F}_t\right] && (30)\\
&= \mathbb{E}\left[\left(IDI_t - e^{-y(t,T)} K e^{-\int_t^T \theta_s \mathrm{d}s}\right)^+ | \mathcal{F}_t\right] && (31)
\end{aligned}$$

where: $y(t, T) := \int_t^T r_s \mathrm{d}s$.

In general, the presence of jumps generate an incomplete market, due to the inability to hedge the continuously distributed jumps. In a way, to perfectly hedge jumps, one requires as many hedging instruments as the cardinality of the jump size distribution. With normally distributed jumps, this requires an uncountably infinite number of hedging instruments. On the other hand in our framework the card$(\mathcal{A})$ is by construction finite. This feature circumscribes our analysis to the standard complete market framework where there is an unique martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$.

Theoretical results and cross-section pricing of interest rate Asian options can be found among others in Geman and Yor (1993) and Chacko and Das (2002). In particular, pricing IDI options were recently studied by Almeida and Vicente (2012) by specifying the overnight rate, r_t as a sum of N processes with $\Theta = 0$ for all N in (13).

When considered the presence of scheduled meetings the pricing of an IDI option can be obtained as:

Proposition 2 *The no-arbitrage price for an European IDI call option is given by:*

$$Call(T,K,R_t) = \mathbb{E}\left[\left(IDI_t - e^{-y(t,T)}Ke^{\int_t^T(\theta_s \mathrm{d}s)}\right)^+ |\mathcal{F}_t\right] \tag{32}$$

$$= \sum_i BS_{call}((r_T|\theta_T = i), \hat{K}_i, T, V(t,T))\mathbb{P}(\theta_T = i) \tag{33}$$

where:

$$BS_{call}((r_T|\theta_T = i), K, T, V(t,T)) = IDI_t N(d_1) - \hat{K}_i P(t,T) N(d_2) \tag{34}$$

with:

$$\hat{K} := Ke^{-\int_t^T \theta_s \mathrm{d}s} \in \mathbb{R}_+ \textit{is the corrected strike price.} \tag{35}$$

$$d_1 = \frac{\log \frac{IDI_t}{KP(t,T)} + \frac{V(t,T)}{2}}{\sqrt{V(t,T)}} \tag{36}$$

$$d_2 = d_1 - \sqrt{V(t,T)} \tag{37}$$

and $V(t,T)$ as in (16).
with $\tau = (T-t)$, $\theta_T := \int_t^T \theta_s \mathrm{d}s \in \mathbb{R}_+$ and $\mathbb{P}(\theta_T = i)$ as (11).

Proof of Proposition 2. Starting with equation (29) we can make explicit the expectations:

$$Call(T,K,r_t) = \int_{\mathcal{A}\times\Omega}\left[\left(IDI_t - e^{-y(t,T)}Ke^{-\int_t^T \theta_s \mathrm{d}s}\right)^+\right]\mathrm{d}F(r_T,\theta_T) \quad (38)$$

$$\int_{\mathcal{A}}\left[\int_{\Omega}\left[\left(IDI_t - e^{-y(t,T)}Ke^{-\int_t^T \theta_s \mathrm{d}s}\right)^+\right]\mathrm{d}F(r_T|\theta_T)\right]\mathrm{d}G(\theta_T) \quad (39)$$

$$\int_{\mathcal{A}}\left[\int_{\Omega}\left[\left(IDI_t - e^{-y(t,T)}\hat{K}\right)^+\right]\mathrm{d}F(r_T|\theta_T)\right]\mathrm{d}G(\theta_T) \quad (40)$$

Conditioning $F(r_T,\theta_T)$ on θ_T the inner integral with the new strike price $\hat{K}$ can be solved as in Almeida and Vicente (2012) but with $V(t,T)$ and $M(t,T)$ adjusted as given in (16) to work with $\Theta \neq 0$. In this step we are solving the same problem as in Alemida and Vicente (2012) but here with modified strike prices $\hat{K}$ which depends explicitly on θ_T. Once solved the earlier step for a given $\theta_T = i$ we repeat step one for every possible θ_T and weighting by its probabilities, $\mathbb{P}(\theta_T = i)$, calculated as equation (11).

■

If $Put(T,K,R_t)$ is the price at time t of the IDI put with strike K and maturity T, then by the put-call parity we state without proof:

Proposition 3 *The no-arbitrage price for an European IDI put option is given by:*

$$Put(T,K,R_t) = \sum_i BS_{put}((r_T|\theta_T = i),K,T,V(t,T))\mathbb{P}(\theta_T = i)$$

where:

$$BS_{put}((r_T|\theta_T = i),K,T,V(t,T)) = KP(t,T)N(-d_2) - IDI_tN(-d_1) \quad (42)$$

with:

$$\hat{K} := Ke^{-\int_t^T \theta_s \mathrm{d}s} \in \mathbb{R}_+ \textit{is the corrected strike price.} \quad (43)$$

$$d_1 = \frac{\log\frac{IDI_t}{KP(t,T)} + \frac{V(t,T)}{2}}{\sqrt{V(t,T)}} \quad (44)$$

$$d_2 = d_1 - \sqrt{V(t,T)} \quad (45)$$

and $V(t,T)$ as in (16).
with $\tau = (T-t)$, $\theta_T := \int_t^T \theta_s \mathrm{d}s \in \mathbb{R}_+$ and $\mathbb{P}(\theta_T = i)$ as (11).

This strategy of conditioning on all possible values of θ_T is conceptually equivalent to Merton (1976) to price option when jumps are presents.

Even though Propositions 2 and 3 were derived for IDI options traded in Brazil these results are valid for any other instruments where the underlying is an overnight rate.

5 Model Calibration

5.1 Simulated monetary decision data

In this section we calibrate the transition matrix using real market prices. But before calibrating the model to real data we performed a Monte Carlo simulation to assess its quality to extract market beliefs about Central Bank decision. We assume different values for the elements of $\mathcal{A}$ for 2 consecutive meetings. The overnight interest rates is described by equation (13). For every set $\mathcal{A}$ we combine all elements to describe futures decision of Central Bank. For instance, if $\mathcal{A} = \{-25bps, 0, +25bps\}$ we have a vector of dimension 9×2 corresponding to all 2-combinations from elements of set $\mathcal{A}$. For every possible combination of monetary decision we use equation (10) and (21) to simulate bond prices at time t and later we solve the optimization problem:

$$\text{argmin } (\hat{B}_t - \check{B}_t)^2 \ s.t: \begin{cases} \sum_j \mathbb{P}(\theta_u = i | \theta_s = j) = 1 \\ \mathbb{P}(\theta_u = i | \theta_s = j) \geq 0, \forall j \end{cases} \tag{46}$$

where $\hat{B}_t$ is obtained by plugging the values of $\mathcal{A}$ into (10) with different values for initial overnight rate r_t. $\check{B}_t$ is the predicted bond price using (21). The first constraint assures that the sum of each line in the transition matrix is equal to 1 and the second constraint assures non-negative values for probabilities. The output from the optimization problem is a vector of dimension 9×2 corresponding to all 2-combinations from elements of set $\mathcal{A}$. Results from the simulation exercise are in tables 1 and 2:

	$1^{st} Meeting$	$2^{nd} Meeting$
$\mathcal{A} = \{-25bps, 0, +25bps\}$	100%	100%
$\mathcal{A} = \{-25bps, 0, +50bps\}$	100%	100%
$\mathcal{A} = \{-50bps, 0, +25bps\}$	100%	99%

Table 1: Calibration exercise for simulated monetary decision. Initial overnight interest rate, $r_t = 10\%$

A similar result is obtained when the overnight interest rate is $r_t = 5\%$:

	$1^{st} Meeting$	$2^{nd} Meeting$
$\mathcal{A} = \{-25bps, 0, +25bps\}$	100%	100%
$\mathcal{A} = \{-25bps, 0, +50bps\}$	100%	100%
$\mathcal{A} = \{-50bps, 0, +25bps\}$	100%	100%

Table 2: Calibration exercise for simulated monetary decision. Initial overnight interest rate, $r_t = 5\%$

We assume that the bond maturity is 4 months and Central Bank Meetings are scheduled every month. Tables 1 and 2 might be read as follows: cell $(2, 2)$ is the percentage of times that the calibration algorithm predicted the right outcome for the first meeting. Cell $(2, 3)$ express the percentage of times that the calibration algorithm predicted the outcomes for the first and second meeting. For the first meeting, probabilities are calculated using equation (12) while for the remaining meeting the probabilities are calculated using equation (11).

Another way to visualize the information above is graphically as depicted below:

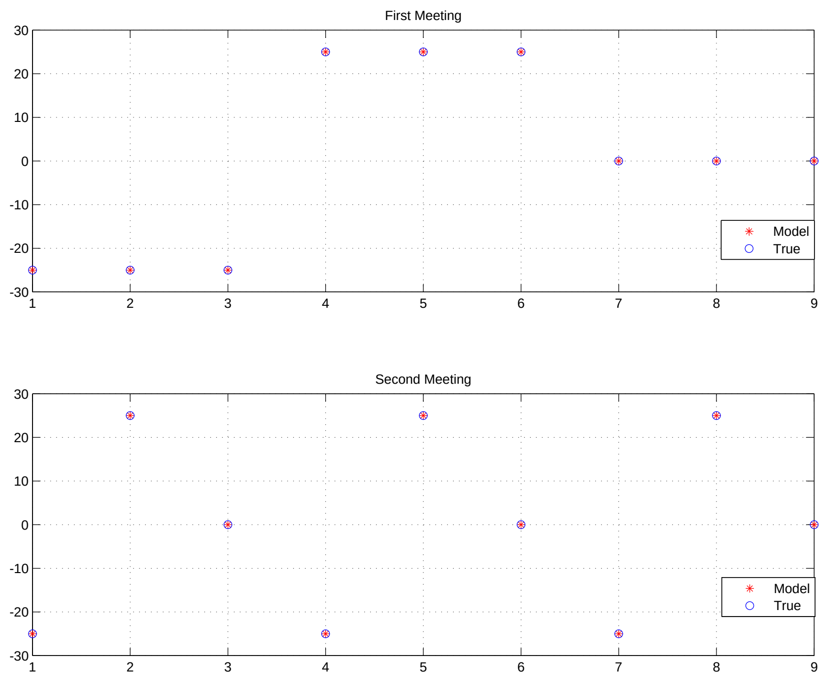

Figure 4: Calibration exercise for simulated monetary decision. Initial overnight interest rate, $r_t = 10\%$

Assuming that $\mathcal{A} = \{-25bps, 0, +25bps\}$ we plotted all combination of two elements of $\mathcal{A}$ representing two possible meetings outcomes, therefore a total of 9 elements. We can read the graph by choosing one element on axes X, for instance, the point 1 at X axes represents two consecutive interest rate reduction, while point 5 is the opposite, two consecutive increases.

We can see that using simulated data the calibration algorithm predicts with high precision outcomes for Central Bank Meetings implied into bond prices. Now we turn to calibrate the model with real market prices.

5.2 Real market prices

We choose to calibrate the model to Brazilian data for two reasons. First, there is a very liquid market for overnight interest rate in Brazil, both for futures and options. Second, Brazil has adopted a Inflation Targeting regime since 1999 with scheduled meeting to define the target rate and interest rate derivatives are used by market participants to bet on future monetary decisions[4]. The overnight interest rate futures[5] traded at BM&FBOVESPA is one of the most liquid short-term interest rate contracts in emerging markets, and the average volume of 1.3 million contracts traded daily is significant even for developed markets. The notional value of the contract is 100,000 BRL (approximately 50,000 USD as of 4/11/2012). DI futures are quoted in terms of rates and are traded in basis-point, but positions are recorded and tracked by the present value of contract, called PU. For a given day t the present value is obtained by discounting the notional value of the contract by the expected overnight interest rate from t up to the day prior to expiration, T. Therefore, at time t we can calculate the present value[6] (PU) of a DI-futures with expiration date of T as:

$$PU_t = \mathbb{E}(e^{-\int_t^T r_s ds} | \mathcal{G}_t) \times 100,000 \tag{47}$$

From equation (47) we verify that the DI futures is very similar to a zero-coupon bond, except that it pays margin adjustments every day. The fact that the contract resembles a zero-coupon bond allows us to use the results derived at earlier sections to extract the implied market transition for $(\theta_t)_{t \geq 0}$ and use them for pricing options. We will calibrate our models as we were in January/2012. We assume that $\mathcal{A} = \{-50bps, 0, +25bps\}$ and we calibrate the model for every day in January to extract the market probabilities of the two next COPOM decisions. The first two COPOM meeting in 2012 were scheduled for January 18 and March 7. Tables below exhibit the transitions matrix implied into DI futures. We do not report all transition matrix due to lack of space, but we do report for 2 days:

Tables 3 and 4 might be read as follows: $\theta = U$ means increase in interest rate; $\theta = D$ means decrease in interest rate; $\theta = N$ means maintenance in interest rate; From tables above we can observe that the transition matrix are quite homogeneous.

[4]The Brazilian Central Bank meeting are called COPOM - Monetary Policy Committee, in Portuguese - and it is conceptually equivalent to FED FOMC meetings. To avoid any potential criticisms about insider information the COPOM releases its decision when the Brazilian market is closed.

[5]Ticker: DI1

[6]In practice, the Brazilian convention for interest rate is exponential compound 252 business day (BD) and margin adjustment are calculated by formula: $PU_t = 100,000/(1+r_t)^{BD/252}$.

	$\theta = U$	$\theta = D$	$\theta = N$
$\theta = U$	0.73	0.13	0.14
$\theta = D$	0.00	0.87	0.13
$\theta = N$	0.33	0.33	0.34

Table 3: Implied transition matrix - 1/2/2012

	$\theta = U$	$\theta = D$	$\theta = N$
$\theta = U$	0.74	0.14	0.12
$\theta = D$	0.00	0.87	0.13
$\theta = N$	0.33	0.33	0.34

Table 4: Implied transition matrix - 1/10/2012

If the purpose of extracting implied probabilities from DI futures is for pricing IDI options we need first determine the marginals probabilities, this is performed using equations (11) and (12), and later use equation (32). The marginal distribution for $\mathcal{A} = \{-50bps, 0, +25bps\}$ are exhibited in figures 5.2 and 5.2:

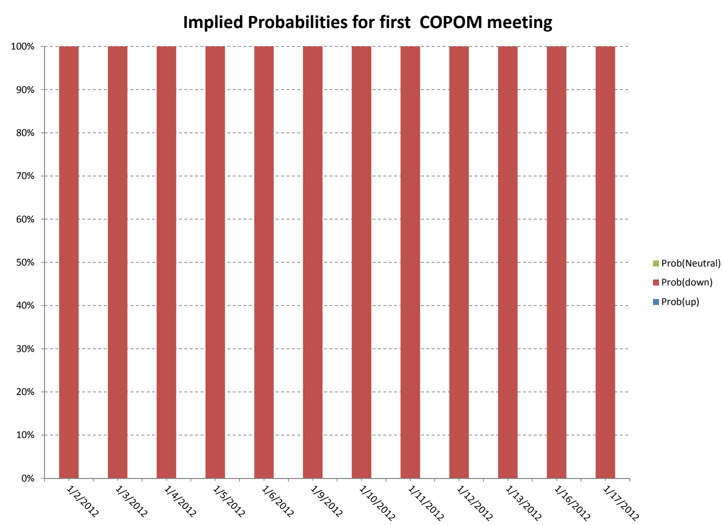

Figure 5: Implied Probabilities for COPOM's decision - Scheduled meeting for 1/18/2012

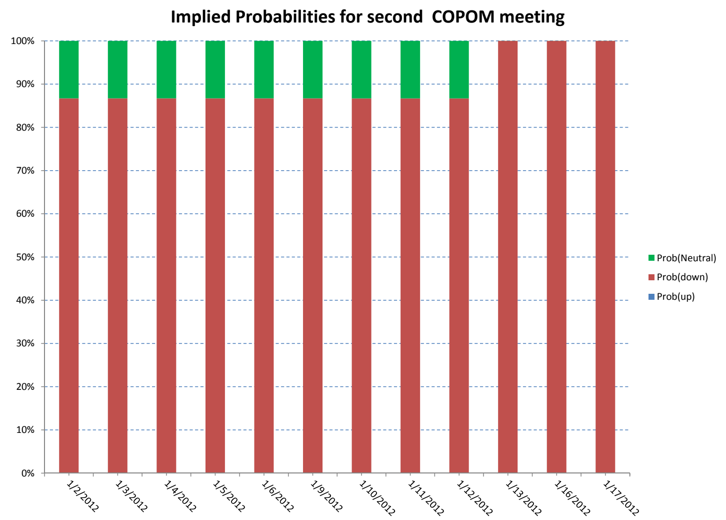

Figure 6: Implied Probabilities for COPOM's decision - Scheduled meeting for 3/7/2012

Ex-post we know that COPOM reduced the target rate by $50Bps$ and $75Bps$ in each meeting. Comparing the results obtained with the model we can assert that market participants could predict the future COPOM decision with high precision. However, this paper is not about efficient ways to predict COPOM's decision per se. It is about how to incorporate market opinions into interest rate derivatives in a consistent way, regardless whether the market can predict future monetary decisions or not.

Regarding asset pricing, a first way to assess the quality of our methodology is by comparing its ability to price DI futures, which are seen as zero-coupon bonds. We compare three models with market prices for a DI futures expiring few days after a scheduled meeting on 02/01/12: pure Vasicek model, Vaiscek with deterministically timed jump - Vasicek TJ, and a naive method using the overnight rate.

According to our methodology where we disentangle the overnight rate evolution from monetary decisions, we need to use a period between two consecutive COPOM meetings because within this interval the observable overnight rate is best described by a continuous process (13).

Since our exercise consists in pricing the DI futures as we were in January/2012 the last between meeting period is from 12/01/2011 to 01/18/2012. To avoid any superposition between calibration and pricing, we calibrated the parameters from (13) along December/2012 using DI futures prices with expiration in 2/1/12 while the pricing step starts on 1/2/12.

After the last COPOM meeting on 11/30/2011, the market sentiment was an additional reduction for the target rate on the next COPOM meeting. In fact, the news on the media were an additional reduction of 50 bps. Therefore putting together the market expectation and calibrated parameters in our model (i.e equation (21)), we are able to price the DI futures and compare its results with a pure Vasicek model and a naive flat forward rate (equation (3) with $\theta = 0$). The results for each models are depicted at figure 7 where we reported the implied rate from each model:

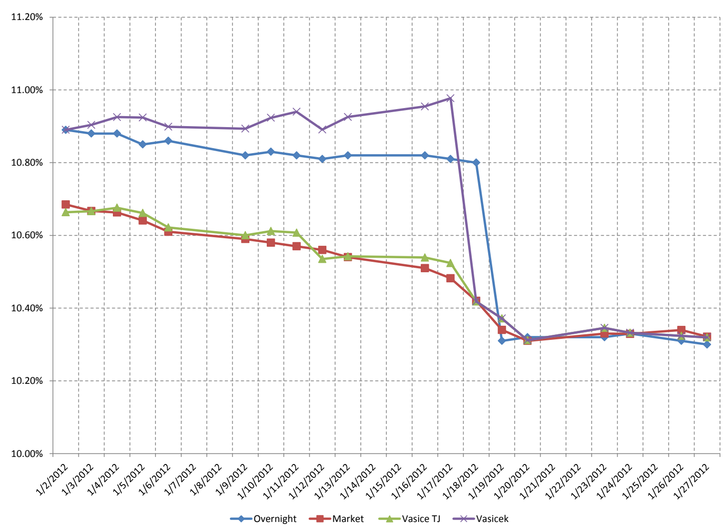

Figure 7: Pricing DI futures - Models performance along January/2012

From the figure above we observe that Vasicek model with deterministic timed jump gives the best performance for pricing DI futures. We observe that pure Vasicek model is very close to the overnight rate and far way from the implied interest rate expected until the DI maturity. On the other hand the Vasicek extended to incorporate deterministic timed jump is very close to market values. The table below quantifies model's performance by mean of mean square errors (MSE):

	Overnight	Vasicek TJ	Vasicek
MSE	4.7E-06	4.46E-08	7.26E-06

Table 5: Mean Square Errors (MSE) for competing models

Table 5 summarized our findings and provides a metric to compare competing models. The Vasicek with deterministic timed jump provides the smallest EQM, in fact its value is 10 times smaller that any other competing model.

Finally we present the results of our methodology applied for pricing IDI options. The parameters driving the continuous process are the same used for pricing DI futures. Probabilities for possible monetary outcome are obtained from IDI futures through the process described in (46). Market Prices for IDI put options are available at BM&FBOVESPA website. We applied our model (equation 32) only for strikes which were traded along each selected day in a 4 weeks horizon in January 2012. All options have expiration date on 4/2/12 and within this period there are two scheduled COPOM meeting.

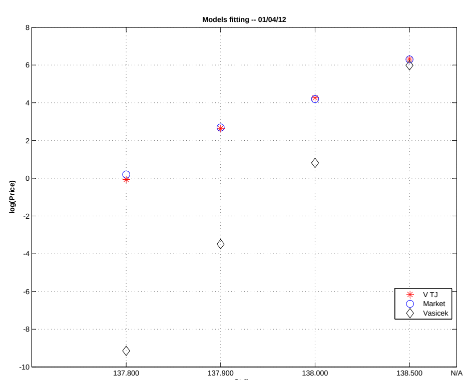

Figure 8: Pricing put IDI options - 4/1/12

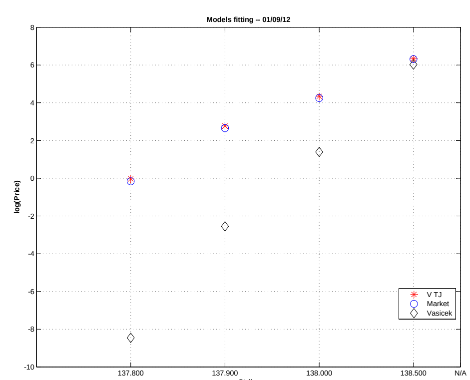

Figure 9: Pricing put IDI options - 9/1/12

Before exhibiting our results some care must be taken because we are handling out-of-the money options with prices inferior to BRL 1 as well as ATM options costing around BRL 550 at same graph. To avoid any scale distortion we decided to plot log-prices. When our model is compared to

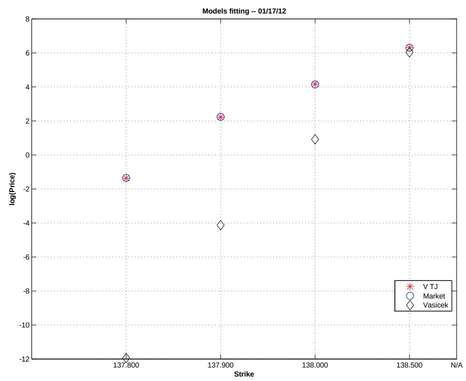

Figure 10: Pricing put IDI options - 17/1/12

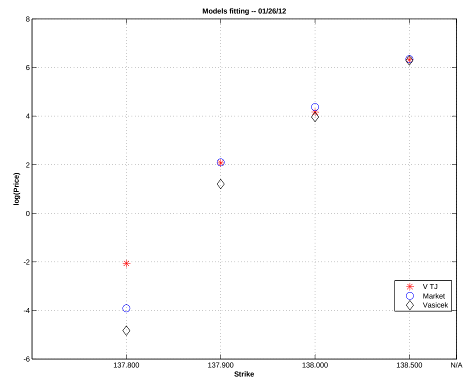

Figure 11: Pricing put IDI options - 26/1/12

market prices we observe a very consistent pattern with those traded into the market. In addition our model outperforms the Vasicek model for every strike.

Finally, we observe that our methodology is flexible enough for modeling equally well out-of-the money and in-the money options without any assumption over the volatility. An important consequence of this framework is the fact that by construction options and ZCB will embed the same probabilities regarding the future monetary policy decisions.

A final remark on our framework concern its comparison to Piazzezi (2005), which also model future monetary outcomes. While option pricing in her framework are obtained using numerical methods to solve a time dependent ODE we only incur in a couple of Black & Scholes-like valuations.

6 Conclusion

Many countries worldwide have adopted Inflation Targeting as a strict rule for conducting their monetary policy. In his turn market participants have tracked carefully all scheduled meeting where the target interest rate is set and trading derivatives to bet on possible outcomes. Standard interest rate models are not suitable for handling deterministic timed events and some level of mispricing is presented when applied for pricing interest rate derivatives. Based on that, we have developed in this paper a stochastic interest rate model able to endogenously incorporate monetary announcements. The model incorporates future monetary decision and therefore allows pricing both futures and options in a consistent way. We calibrate the model to data from Brazil. Brazil came up with the right place to apply our model because it has adopted an inflation targeting regime since 1999 and there is a very liquid overnight interest rate derivatives market which are used by market participants to bet on future monetary decisions. When compared to market prices the model provided good performance and outperformed the standard Vasicek model for pricing liquid put options. Although the model was applied to Brazilian data it can be used in other jurisdictions which announce their policy decisions at regularly scheduled meetings such as England, Australia and even the US.

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