UNIVERSIDADE FEDERAL DO RIO DE JANEIRO
INSTITUTO COPPEAD DE ADMINISTRAÇÃO

MARCELO LEWIN

Portfolio Strategies in the Presence of Multiple Regimes

RIO DE JANEIRO

2023

MARCELO LEWIN

Portfolio Strategies in the Presence of Multiple Regimes

Tese de Doutorado apresentada ao Instituto COPPEAD de Administração, da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Doutor em Administração.

Orientador: Prof.º Carlos Heitor Campani, Ph.D.

RIO DE JANEIRO

2023

CIP - Catalogação na Publicação

Lewin, Marcelo

Portfolio Strategies in the Presence of Multiple Regimes / Marcelo Lewin. – Rio de Janeiro, 2023.
125 f.

Orientador: Carlos Heitor d'Àvila Pereira Campani.

Tese (doutorado) – Universidade Federal do Rio de Janeiro, Instituto COPPEAD de Administração, Programa de Pós-Graduação em Administração, 2023.

1. Regime switching models. 2. Dynamic asset allocation. 3. Analytical solutions. 4. Constrained portfolios. 5. risk factors. I. Campani, Carlos Heitor, orient. II. Título.

Elaborado pelo Sistema de Geração Automática da UFRJ com os dados fornecidos pelo(a) autor(a), sob a responsabilidade de Miguel Romeu Amorim Neto - CRB-7/6283.

Marcelo Lewin

Portfolio Strategies in the Presence of Multiple Regimes

Tese de Doutorado apresentada ao Instituto COPPEAD de Administração, da Universidade Federal do Rio de Janeiro, como parte dos requisitos necessários à obtenção do título de Doutor em Administração.

Aprovada em 15/02/2023 por:

____

Prof. Carlos Heitor d'Ávila Pereira Campani, Ph.D. (COPPEAD/UFRJ)

____

Prof. Ricardo Pereira Câmara Leal, D.Sc. (COPPEAD/UFRJ)

____

Prof. Gustavo Silva Araújo, D.Sc. (EPGE/FGV)

____

Prof. Jefferson Augusto Colombo, D.Sc. (EESP/FGV)

____

Prof. Raphael Moses Roquete, D.Sc. (FACC/UFRJ)

RIO DE JANEIRO

2023

ACKNOWLEDGEMNTS

I understand that we are never alone. Therefore, any project is a team effort. Thus, I thank the company of everyone on Earth and in Heaven who is part of this journey.

I express my deepest gratitude to my family, who have taught and encouraged me throughout life. So many times, you have strengthened me, even when I did not realize it right away. Gratitude extends to my friends, from whom I learned the beauty of balance in life. Your wisdom has inspired me in my academic and professional choices. Words will never be enough to express my gratitude to each and every one of you.

On behalf of all my professors, I would like to thank my advisor, whom I consider a friend, for trusting my dedication. I am grateful for so many real options.

I also thank the financial support from the following institutions to this Thesis (in alphabetical order): ANBIMA – Brazilian Financial and Capital Markets Association, Brasilprev Research Chair, CNPq – Conselho Nacional de Desenvolvimento Científico e Tecnológico, and FAPERJ – Fundação Carlos Chagas Filho de Amparo à Pesquisa do Estado do Rio de Janeiro.

Finally, I extend thanks to the COPPEAD staff, faculty, and colleagues. I have been in the institution for six years, counting my Master's and Doctorate. Over these years, COPPEAD has offered me something that I have always believed in: the education that transforms. Having this opportunity is unforgettable.

All of you contributed to this Thesis. Thank you from the bottom of my heart.

ABSTRACT

The Thesis comprises three articles where we advance with the model from Campani, Garcia, and Lewin (2021), the CGL model, so that professional investors can apply it in practice. The model is the only solution using a regime-switching framework with the stochastic recursive utility, a state-of-the-art function, for optimizing portfolio strategies. The articles test the CGL model with different configurations to assess its performance and robustness in different markets, asset classes, and constraining policies. In the first article, we identify three economic regimes in the Brazilian market, with a portfolio formed with stocks and bonds. Then, the second article studies the US stock market, where we identify four regimes with a size-based portfolio. It also introduces two procedures to constrain the CGL strategies: a maximum leverage control and a low-turnover control. The first calculates dynamic adjustments to the risk aversion parameter to constrain the leverage indicated by optimal strategies. The second sets a dynamic threshold to trim minor rebalancing, reducing portfolio turnover and mitigating costs. Finally, the third article creates a portfolio of Brazilian stocks based on Fama and French (1993) and Carhart's (1997) four-factor model. This article closes a literature gap by using the CGL model to allocate such a portfolio: it is the first strategy with risk-factors based on regimes and the recursive utility function. The three articles evaluate the results in out-of-sample exercises, which consistently indicate that the CGL model outperforms the market benchmarks in the long run. Hence, the Thesis extends paths to fund managers who aim to adopt this methodology to benefit from sophisticated dynamic strategies.

KEYWORDS: regime switching models; dynamic asset allocation; analytical solutions; constrained portfolios; risk factors.

LIST OF FIGURES

LIST OF TABLES

CONTENTS

1. INTRODUCTION

This Doctoral Thesis consists of three articles. They integrate the asset allocation research line based on regime-switching models using the stochastic differential recursive utility function. The research began during Professor Carlos Heitor Campani's Doctorate in 2009 at Edhec (France), with the supervision of Professor René Garcia. After 2017, then supervised by Professor Carlos Heitor Campani, Marcelo Lewin dedicated his Master's and Doctorate at COPPEAD/UFRJ to the continuity of the research. With this development, Marcelo Lewin authored or co-authored six of the seven papers of the research line, where the framework is named the CGL model, referring to (Carlos Heitor) Campani, (René) Garcia, and (Marcelo) Lewin.

There are two primary pillars in the CGL model: the premise that asset returns are governed by more than one economic regime and the utility function applied to optimize the allocation strategy. First, assuming that regimes are unobservable economic states, the CGL model infers them through a Markovian probabilistic system. It considers the asset return processes that compose the investor's portfolio to infer the probabilities. Second, the CGL model uses the stochastic differential utility function from Duffie and Epstein (1992), which, as Epstein and Zin's (1989), belongs to the class of recursive utility functions. Munk (2013) demonstrates that portfolio optimization strategies using such a class more realistically meet the investor's risk preferences than the existing functions so far, such as the traditional power utility functions (as CRRA and HARA) and the well-known Markowitz's quadratic function (and its efficient frontier).

Therefore, based on the asset returns, we estimate a fully quantitative model and the probabilities of each regime occurrence. The CGL model was introduced in Campani, Garcia, and Lewin (2021). It is the first (and only) model allocating assets with multiple regimes using the stochastic recursive utility function to define investor preferences.

The Thesis contribution is improving such a model so professional investors can apply it in practice. Throughout it, we test the CGL model's performance and robustness under different configuration sets, present new controls to constrain the model's optimal strategies, and introduce the CGL model with the risk-factors model from Fama and French (1993) and Carhart (1997). This Thesis was awarded with the XV ANBIMA Prize in recognition of the best Doctoral project in capital markets in Brazil in 2019.

Below, we summarize the three articles of the Thesis.

Article 1. "Portfolio Management under Multiple Regimes: Out-of-Sample Performance in the Brazilian Market" by Marcelo Lewin and Carlos Heitor Campani: Revista Brasileira de Finanças (RBFin), Vol. 18, No. 3, 52 – 79, 2020 (published in Portuguese).[1][2]

The article presents the out-of-sample results of applying the CGL model. It is the first of the research line to perform such exercise, reflecting the situation that a fund manager would find when realistically using the CGL model. We identified three regimes in the returns from Ibovespa, IMA-B 5+, IMA-B 5, and CDI. The CGL model statistically outperformed the benchmarks, which stimulated us to continue with the research.

Article 2. "Constrained Portfolio Strategies in a Regime-Switching Economy" by Marcelo Lewin and Carlos Heitor Campani: paper accepted and forthcoming at the Financial Markets and Portfolio Management (FMPM), 2022.[3]

The second article of the Thesis introduces two new advances to the CGL model: a leverage-constraining method and a rebalancing filter. The first creates optimal constrains to the strategies, dynamically mitigating volatility. The second smooths turnover with a dynamically adjusted threshold. We tested both in an out-of-sample exercise, net of transaction costs (the last step towards absolute realism), with US equities. Such advances enabled to identify CGL portfolios with lower volatility and higher Sharpe ratios than conventionally constrained benchmarks. Moreover, the certainty equivalent returns indicate that, in this exercise, constrained

[1] Article 1 was published in Portuguese with the title in two languages. The Portuguese version of the title is "Gestão de Carteiras sob Múltiplos Regimes: Performance Fora da Amostra no Mercado Brasileiro".
[2] Link to article 1: http://bibliotecadigital.fgv.br/ojs/index.php/rbfin/article/view/81210
[3] Link to article 2: https://link.springer.com/article/10.1007/s11408-022-00414-x.

CGL models outperformed both single-state models and unconstrained regime-switching models with statistical significance.

Article 3. “Optimal Constrained Strategies for Factor-Based Investing in the Brazilian Stock Market” by Marcelo Lewin and Carlos Heitor Campani: working paper, 2023.

The last article introduces the CGL model with Fama and French's (1993) model. Thus, we present an original application of a factor-based portfolio using a regime-switching model with the recursive stochastic utility function. First, we identified Cahart's (1997) four risk factors in the Brazilian stock market: SMB (small minus big), HML (high minus low), and MOM (momentum), in addition to the market factor. Then, we conducted an out-of-sample exercise, net of transaction costs, to benchmark the CGL model with passive and active investment strategies. The CGL model's Sharpe ratio outperformed the competitors' in the complete exercise and in shorter subsamples of the out-of-sample exercise. In addition, the CGL portfolio without any leverage offered the lowest volatility relative to the benchmarks (but with competitive returns). The certainty equivalent returns show that the CGL model outperformed competitors with statistical significance.

The Thesis demonstrates that regime-switching models using the recursive utility function provides competitive strategies relative to benchmarks. Moreover, it highlights that fund managers who plan to adopt this methodology will benefit from sophisticated dynamic strategies, as the results indicate that the CGL model outperforms its market benchmarks in the long run.

2. GESTÃO DE CARTEIRAS SOB MÚLTIPLOS REGIMES: PERFORMANCE FORA DA AMOSTRA NO MERCADO BRASILEIRO[4][5]

2.1 RESUMO

Propomos uma estratégia de alocação dinâmica para um investidor que considera três regimes econômicos latentes, estimados através dos retornos de cinco classes de ativos brasileiros. A estratégia é definida a partir de uma solução analítica aproximada numa configuração econômica realista, porém a ponto de não permitir solução analítica exata. Testamos a performance da carteira fora da amostra e o resultado superou o retorno de todos os benchmarks analisados em 6 dos 10 anos em tela, tendo apresentado média de retorno semanal superior a qualquer dos benchmarks com significância estatística. De 2010 a 2019, nossa estratégia alcançou um retorno médio igual a 21,6% ao ano contra, por exemplo, 9,8% a.a. do CDI e 4,7% a.a. do Ibovespa. Não obstante, com retornos médios significativamente superiores, os resultados da comparação desta carteira com uma carteira de referência igualmente ponderada indicam a importância do modelo proposto à luz da gestão ativa de portfólios. Por fim, a comparação com uma carteira onde o investidor considera um único regime deixa claro a importância dos múltiplos regimes na estratégia de alocação. Concluímos que a solução proposta pode interessar a gestores de investimentos e objetivamos fomentar a aplicação de modelos de múltiplos regimes no mercado brasileiro de gestão de carteiras de investimentos.

[4] This section presents the first article of the Thesis. It is in Portuguese to exhibit the article as published.

[5] Artigo publicado na Revista Brasileira de Finanças (RBFin), Vol. 18, No. 3, 52 – 79, 2020. https://bibliotecadigital.fgv.br/ojs/index.php/rbfin/article/view/81210/78264

2.2 INTRODUÇÃO

Evidências empíricas sugerem que modelos econômicos que preveem mudanças de regimes (*regime switching models*) acrescentam relevante informação à gestão de carteiras de investimento, estimando em tempo real as probabilidades de ocorrência de regimes em períodos vindouros que modelam os retornos, volatilidades e intercorrelações de ativos, como apontam Ang e Bekaert (2004), Tu (2010), Guidolin e Hyde (2012) e Ang e Timmermann (2012).

A revisão da literatura feita por Guidolin (2011) sugere que bastariam dois regimes para descrever a distribuição dos retornos de ações e intercorrelações; e apenas três regimes para descrever o mercado de renda fixa. De forma genérica, porém, a coerência entre regimes com diferentes classes de ativos e/ou regiões geográficas pode não ser perfeita, segundo Bae et al. (2014) e Case et al. (2014). Logo, uma economia com classes de investimento diversificadas pode se caracterizar por processos de mudanças de regimes não-perfeitamente sincronizados ou que apresentem outro número de regimes, como Guidolin e Timmermann (2007) que estimaram quatro regimes nos EUA a partir dos ativos: renda fixa de longo prazo, dois portfólios de ações (*large* e *small caps*) e a taxa livre de risco.

A contribuição desta pesquisa à literatura advém da estimação dos regimes a partir de um conjunto relevante e original de cinco classes de ativos com o viés do investidor brasileiro, e a avaliação de estratégia de alocação baseada no modelo de Campani, Garcia e Lewin (2021) – doravante modelo CGL. Trabalhamos com as seguintes classes de ativos: *cash* (taxa livre de risco no Brasil), *carry-trade* (taxa livre de risco nos EUA convertida em reais), renda fixa de curto e longo prazo, e renda variável. O objetivo principal da pesquisa foi avaliar a performance do modelo CGL fora da amostra com um conjunto original de classes de ativos, estendendo os testes de robustez realizados por Lewin e Campani (2020) – estes com a limitação de terem sido dentro da amostra.

Como resultado primário, apesar de utilizarmos classes de investimento diversificadas como Guidolin e Timmermann (2007) e Lewin e Campani (2020), vemos que a influência das classes de renda fixa em nosso portfólio indica a estimação de apenas três regimes na aplicação do modelo, como sugere a literatura apresentada acima em referência ao mercado de renda fixa.

Como resultado principal, ao aplicarmos os parâmetros dos múltiplos regimes e o modelo de alocação CGL, tal estratégia apresenta um retorno médio semanal estatisticamente superior a todos *benchmarks* considerados nesta pesquisa no período analisado (2010 a 2019). Além disso, a carteira CGL logrou êxito ao obter o maior retorno dentre todos os *benchmarks* em 6 dos 10 anos analisados. Assim, concluímos que o modelo apresenta resultados empíricos que podem despertar o interesse de gestores de carteiras e de fundos de investimento no mercado.

2.3 REVISÃO DE LITERATURA

A pesquisa seminal de Hamilton (1989) apresenta o modelo econométrico de mudanças de regimes onde a previsão dos estados econômicos (regimes) é realizada através de um sistema probabilístico de cadeias de Markov, originado a partir dos dados extraídos nos retornos dos ativos. Ang e Bekaert (2002) e Graflund e Nilsson (2003) foram as primeiras pesquisas a aplicar esta técnica na alocação de ativos. Ambas as pesquisas identificaram dois regimes (*bull* e *bear market*), porém ignoraram a característica latente dos regimes, considerando-os totalmente observáveis. Posteriormente, Guidolin e Timmermann (2007) identificaram a existência de quatro regimes latentes e, utilizando a função potência de utilidade (*power utility function*), otimizaram carteiras de investimento através da simulação de Monte Carlo. Esta clássica função utilidade permite um tratamento matemático simplificado que poupa tempo na simulação, entretanto se distancia da realidade empírica ao prever uma relação injustificável entre os parâmetros de risco de alocação (γ) e de consumo intertemporal (ψ). Tal relação é apresentada por Munk (2013).

Campani, Garcia e Lewin (2021) igualmente identificaram quatro regimes latentes aplicando o modelo de mudança de regimes sob a mesma configuração de ativos de Guidolin e Timmermann (2007). Porém, o modelo CGL considera, para efeitos de alocação de ativos, a função de utilidade estocástica diferencial de Duffie e Epstein (1992) – uma classe dentre as funções recursivas de utilidade, que devidamente separa os parâmetros de risco γ e ψ.

Infelizmente, modelos realistas para seleção de carteiras ainda não contam com soluções analíticas exatas em economias com múltiplos regimes. Em consequência, a simulação de Monte Carlo (SMC) surge como opção natural: entretanto pode facilmente tornar-se inviável por conta do tempo de processamento. Isto abriu caminho para Campani, Garcia e Lewin (2021) proporem uma solução analítica aproximada, baseada na aproximação proposta por Campani e Garcia (2019) em uma economia de único regime e função de utilidade estocástica diferencial: os autores demonstraram que a aproximação é suficientemente precisa para a alocação de ativos. Uma solução analítica permite a avaliação das estratégias de alocação de forma mais profunda – por exemplo: o impacto dos parâmetros de mercado pode ser verificado de maneira mais nítida – e superam o severo custo de tempo imposto pela SMC em configurações realistas.

O modelo CGL já foi aplicado no Brasil por Lewin e Campani (2020), que identificaram quatro regimes latentes na economia em quatro classes de ativos: renda fixa, renda variável nacional e renda variável internacional (convertida em reais), além da taxa livre de risco. Os autores mostram que os resultados da carteira CGL foram promissores, porém a análise foi feita dentro da amostra. A atual pesquisa inova ao avaliar a performance do modelo CGL fora da amostra, afinal "no trabalho aplicado, modelos dinâmicos de séries temporais serão apenas tão bons quanto sua performance preditiva", ou seja, fora da amostra – Guidolin (2011).

Os regimes e a carteira são estimados a partir de cinco classes de ativos: taxa livre de risco, *carry-trade* sob a ótica de um investidor brasileiro (taxa livre de risco dos EUA convertida em reais), renda fixa de longo e de curto prazo, e renda variável. A estratégia de alocação se mostrou um pouco mais complexa pelo maior número de ativos em relação a Lewin e Campani (2020), e devido ao aumento da participação das classes de renda fixa na carteira.

Sob esta configuração, identificamos três regimes latentes e propomos a estratégia do modelo CGL com vendas a descoberto e horizonte finito. Mostraremos que os resultados são relevantes em relação aos principais *benchmarks* brasileiros. Com a estimação fora da amostra, os resultados podem também ser obtidos por gestores de ativos e a ambição deste trabalho é exatamente oferecer a aplicação de um modelo pioneiro para gerir carteiras no mercado brasileiro de forma mais eficaz, sob a ótica de um modelo de múltiplos regimes econômicos.

2.4 METODOLOGIA E DADOS

2.4.1 PRÊMIOS DE RISCO

Acompanhando os desenvolvimentos econométricos descritos em Campani, Garcia e Lewin (2021) e Lewin e Campani (2020), configuramos um modelo de tempo contínuo, discretizado para efeito prático em semanas – o que será justificado adiante. Consideramos um mercado eficiente e sem saltos, incorporando $n + 1$ classes de ativos: um ativo de curto prazo livre de risco e n ativos de risco. Utilizamos prêmios de risco para absorver a volatilidade do ativo livre de risco nos retornos dos ativos de risco. Calculamos os retornos em excesso ($\hat{r}_{i,t+1}$), também denotados como prêmio de risco, de t para $t + 1$, com t medido em semanas:

$$\hat{r}_{i,t+1} = \frac{1 + r_{i,t+1}}{1 + rf_{t+1}} - 1, \text{ com} \tag{1a}$$

$$r_{i,t+1} = \frac{S_{i,t+1} - S_{i,t}}{S_{i,t}}, \tag{1b}$$

onde $i = 1, 2, \dots, n$; $S_{i,t}$ é preço do ativo de risco i na semana t, o qual incorpora dividendos e quaisquer outros efeitos de retorno; $r_{i,t+1}$ é o retorno total do ativo de risco i; rf_{t+1} é o retorno do ativo livre de risco na semana de t a $t + 1$.

2.4.2 CONFIGURAÇÃO DA ECONOMIA E ESTIMAÇÃO DOS PARÂMETROS

Consideramos que a dinâmica dos prêmios de risco dos n ativos pode ser definida pelo seguinte processo estocástico multidimensional:

$$\begin{bmatrix} d\hat{r}_{1,t} \\ d\hat{r}_{2,t} \\ \cdots \\ d\hat{r}_{n,t} \end{bmatrix} = \begin{bmatrix} \mu_{1,t} \\ \mu_{2,t} \\ \cdots \\ \mu_{n,t} \end{bmatrix} dt + \begin{bmatrix} \sigma_{11,t} & 0 & \dots & 0 \\ \sigma_{21,t} & \sigma_{21,t} & \dots & 0 \\ \cdots & \cdots & \dots & \cdots \\ \sigma_{n1,t} & \sigma_{n2,t} & \dots & \sigma_{nn,t} \end{bmatrix} \begin{bmatrix} dZ_{1,t} \\ dZ_{2,t} \\ \cdots \\ dZ_{n,t} \end{bmatrix}, \tag{2a}$$

com as seguintes definições convenientes:

$$\boldsymbol{\mu}_{s,t} = \begin{bmatrix} \mu_{1,t} \\ \mu_{2,t} \\ \dots \\ \mu_{n,t} \end{bmatrix}, \quad \boldsymbol{\sigma}_{s,t} = \begin{bmatrix} \sigma_{11,t} & 0 & \dots & 0 \\ \sigma_{21,t} & \sigma_{21,t} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \sigma_{n1,t} & \sigma_{n2,t} & \dots & \sigma_{nn,t} \end{bmatrix} \quad e \quad \boldsymbol{dZ_t} = \begin{bmatrix} dZ_{1,t} \\ dZ_{2,t} \\ \dots \\ dZ_{n,t} \end{bmatrix}, \tag{2b}$$

onde $\boldsymbol{\mu}_{s,t}$ corresponde ao vetor coluna $(n\ x\ 1)$ contendo os prêmios de risco instantâneos esperados (*drifts*); $\boldsymbol{\sigma}_{s,t}$ corresponde à matriz de volatilidades $(n\ x\ n)$ desenhada como uma matriz triangular inferior (sem absolutamente nenhuma perda de generalidade) e $\boldsymbol{dZ_t}$ corresponde ao vetor coluna $(n\ x\ 1)$ contendo n incrementos de processos de Wiener padronizados e independentes entre si.

O vetor dos prêmios de risco ($\boldsymbol{\mu}_{s,t}$) e a matriz de volatilidades ($\boldsymbol{\sigma}_{s,t}$) variam no tempo e em função da variável de estado Y_t (não-observável). Definimos que a variável de estado Y_t refere-se à transição entre os regimes como um processo de cadeias de Markov independente, em tempo contínuo, que é contínua para o limite da direita e admite apenas valores em $R = \{1, 2, \dots, m\}$ onde R corresponde ao conjunto finito de m possíveis regimes econômicos. No estado $Y_t = i,\ i \in R$, temos:

$$\boldsymbol{\mu}_{s,t} = \begin{bmatrix} a_{1,i} \\ a_{2,i} \\ \dots \\ a_{n,i} \end{bmatrix} \quad e \quad \boldsymbol{\sigma}_{s,t} = \boldsymbol{\sigma}_{s,i} = \begin{bmatrix} \sigma_{11,i} & 0 & \dots & 0 \\ \sigma_{21,i} & \sigma_{21,i} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \sigma_{n1,i} & \sigma_{n2,i} & \dots & \sigma_{nn,i} \end{bmatrix}, \tag{3}$$

com coeficientes $a_{j,i}$ e matrizes $\boldsymbol{\sigma}_{s,i}$ constantes para cada $i \in R$ e $j = 1, 2, \dots, n$. Ressaltamos que $\boldsymbol{\sigma}_{s,i}\boldsymbol{\sigma}_{s,i}^T$ representa a matriz de variância-covariância dos retornos dos ativos sob o regime i. Todos os elementos de $\boldsymbol{\sigma}_{s,i}$ são definidos como volatilidades parciais: por exemplo, $\sigma_{21,i}$ representa a volatilidade parcial do ativo 2 com relação ao primeiro processo de Wiener ($dZ_{1,t}$) no regime i.

Os *drifts* dos prêmios de risco dependem dos regimes e simultaneamente variam no tempo, mesmo que o regime permaneça inalterado. Eles são coletados na matriz de *drifts* (que varia no tempo) $\boldsymbol{D_{s,t}}$ cujas dimensões são $n\ x\ m$:

$$\boldsymbol{D_{s,t}} = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \dots & \dots & \dots & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,m} \end{bmatrix}. \qquad (4)$$

Voltando à configuração da variável de estado Y_t , ela governa os estados da economia e não é observável. O processo pode começar em qualquer momento t_0 sob qualquer regime, que Y_t permanecerá no mesmo regime por um período de tempo exponencialmente distribuído até saltar para outro regime. A permanência e os saltos entre regimes são tratados através de probabilidades de transição. Considerando que o regime vigente é i, a probabilidade de saltar para outro regime j no período de tempo Δt é dada por $P_{ij,\Delta t} = \frac{\lambda_{ij}}{\sum_{k\neq i}\lambda_{ik}}(1 - e^{-\sum_{k\neq i}\lambda_{ik}\Delta t})$, onde $j \neq i\ \in R$ e $\lambda_{ij} > 0$. Definimos $\lambda_{ii} = -\sum_{k\neq i}\lambda_{ik} \leq 0$ de forma que $P_{ij,\Delta t} = \frac{\lambda_{ij}}{-\lambda_{ii}}\left(1 - e^{\lambda_{ii}\Delta t}\right)$. Assim, a probabilidade de permanecer no mesmo regime i no próximo período de tempo Δt será $P_{ii,\Delta t} = e^{\lambda_{ii}\Delta t}$. Sob a premissa de que sejam constantes, os parâmetros λ_{ij} representam a densidade da probabilidade de transição entre regimes.

Com base nestes processos, estimamos os parâmetros do modelo de regimes pelo método da máxima verossimilhança (MLE – *Maximum Likelihood Estimation*) de acordo com o Hamilton (1989), onde os parâmetros estimados são: os elementos da matriz $\boldsymbol{D_{s,t}}$ referente aos *drifts* dos ativos por regime; os elementos da matriz de volatilidades por regime $\boldsymbol{\sigma_{s,i}}$; e as probabilidades de transição entre os regimes $P_{ij,\Delta t}$ (onde $j \neq i\ \in R$). Na próxima seção, apresentaremos os valores de cada parâmetro estimado. Aqui ressaltamos que o total de parâmetros desta aplicação pode ser calculado como: $m * n + m * (n + 1) * n * 0,5 + m * (m - 1)$. Logo, sob $m = 3$ regimes e $n = 4$ ativos de risco, esta configuração demandou a estimação de 48 parâmetros.

Guidolin e Ono (2006) apontam que um modelo parcimonioso em termos de número de observações e regimes apresenta um índice de saturação superior à 26. Esta medida representa a quantidade de observações disponíveis por parâmetro estimado e é obtida através do número de ativos (n) multiplicado pela quantidade de observações dividido pelo total de parâmetros.

Como nossa pesquisa objetivava a realização de *backtests* para calcular a performance com estimações fora da amostra, um número alto de regimes poderia inviabilizar o projeto pois a série restritiva se estendia apenas até set/2003. Para encontrar o número ótimo de regimes e definir o período de análise realizamos 563 otimizações (MLE) acompanhando a restrição imposta pelo índice de saturação. Testamos modelos com $m = 3, 4, 5\ e\ 6$ regimes sob diversas janelas de observação, com dados mensais, semanais e diários. Avaliamos todos os resultados através dos parâmetros de *output* da otimização, como: *f-value*, *exit flag* e número de iterações.

A estimação mais robusta consistentemente ocorria sob $m = 3$ regimes e dados semanais. Com o índice de saturação superior às demais configurações, nesta, os parâmetros estimados não se alteravam significativamente nos testes de *stress* aos quais submetemos o modelo ao variar as janelas de dados. Em contrapartida, a série com dados mensais encurtava sobremaneira o número de observações, e a série diária não agregava melhora nos indicadores da estimação.

Ao rodarmos estimações com $m = 4,\ 5$ e 6, encontramos sistematicamente 3 regimes “ativos” (com características similares ao modelo $m = 3$) mas de acordo com as probabilidades filtradas mais regimes praticamente não ocorriam. As probabilidades filtradas, sendo aquelas que indicam o regime de $t+1$ em t, mostravam valores significativos apenas para 3 regimes ao passo que nos demais regimes tais valores tendiam à zero. Na seção a seguir discutiremos características especificas da carteira que favoreceram à estimação de “apenas” 3 regimes.

Não obstante, o índice de saturação também foi utilizado para estabelecermos o início das janelas como a primeira data disponível (set/2003) e aumentarmos gradativamente o total de observações. Desse modo, começamos com um índice de saturação de 27 na primeira janela (até dez/2009) e alcançarmos o índice de saturação de 71 na última janela (até dez/2019).

2.4.3 CONFIGURAÇÃO DO MODELO DE ALOCAÇÃO

Para realizar a estratégia de alocação, configuramos um investidor que maximiza o valor esperado da utilidade de seus investimentos em cada momento t através da função recursiva de utilidade estocástica diferencial de Duffie e Epstein (1992) – que se destaca da função Epstein-Zin por operar no campo de contínuo (Munk, 2013):

$$J_t = E_t\left[\int_{u=t}^{T} f(C_u,J_u)\,du + \frac{W_T^{1-\gamma}}{1-\gamma}\right], \tag{5}$$

onde E_t corresponde ao valor esperado condicional à data t; T é o horizonte do investimento e t, o instante atual; a função f é o agregador de recursividade da função utilidade (J_t), dado em função de C_u que denota a taxa de consumo no momento u que satisfaz o investidor, e J_u representando a utilidade continuada em u; por sua vez, W_T representa a riqueza do investidor no horizonte de investimento T e γ, seu parâmetro de aversão a risco. Na equação (6) abaixo, detalha-se o agregador da função utilidade (descrito de forma genérica em função de duas variáveis C e J):

$$f(C,J) = \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)\,J\left\{\left[\frac{C}{[(1-\gamma)\,J]^{\frac{1}{1-\gamma}}}\right]^{1-\frac{1}{\psi}} - 1\right\}, \tag{6}$$

onde β é a taxa de preferência no tempo; e ψ é a elasticidade intertemporal de substituição, *i.e.*, parâmetro de risco de consumo intertemporal. Consideramos $\psi = \infty$ a fim de eliminar o consumo e restringir o problema da pesquisa à alocação dos ativos. Seguindo Campani, Garcia e Lewin (2021) e Guidolin e Timmermann (2007), consideramos $\gamma = 5$ e $\beta = 2\%$ ao ano. Na fórmula acima, C representa a taxa de consumo instantânea e J, a função utilidade (note-se a recursividade da função de utilidade, pois esta depende de si mesma nos momentos futuros).

Assumindo regimes latentes, ou seja, não observáveis, os investidores estimam probabilidades de ocorrência de cada regime no período seguinte, para o qual precisam tomar a decisão de alocação, em um vetor unidimensional ($\boldsymbol{\pi_t}$). Como demonstram Campani, Garcia e Lewin (2021), a solução geral que quantifica a utilidade total otimizada ($V_t = \sup J_t$) do investidor a cada momento (t) admite a solução separável da riqueza (W_t) no seguinte formato:

$$V(W_t, \boldsymbol{\pi_t}, \tau) = H(\boldsymbol{\pi_t}, \tau)\,\frac{W_t^{1-\gamma}}{1-\gamma}\,, \tag{7}$$

onde $\tau = T - t$ é o tempo restante até o horizonte final de tempo e $H(\boldsymbol{\pi_t}, \tau)$ é função deste mesmo horizonte e do vetor de probabilidades dos regimes. Uma expressão analítica exata para $H(\boldsymbol{\pi_t}, \tau)$ ainda não foi encontrada na literatura e, portanto, Campani, Garcia e Lewin (2021) propõem uma expressão analítica aproximada para esta função. Eles mostram que tal aproximação é suficientemente precisa para alcançar a alocação ótima de portfólio com a maximização da utilidade. A função H pode então ser aproximada da seguinte forma:

$$H(\boldsymbol{\pi_t}, \tau) = exp\left[A_0(\tau) + \sum_{i=1}^{m} A_i(\tau)\, \pi_{i,t} + \sum_{i=1}^{m} B_i(\tau)\pi_{i,t}^2 + \sum_{j \neq i} C_{ij}(\tau)\, \pi_{i,t}\, \pi_{j,t}\right]. \tag{8}$$

Com a solução aproximada para $V(W_t, \pi_t, \tau)$, é possível resolver a equação de Bellman equivalente ao problema de otimização em tela para se obter os coeficientes $A_0, A_i, B_i \; e \; C_{ij}$ da função $H(\boldsymbol{\pi_t}, \tau)$, todos eles dependentes do horizonte τ de investimento. O documento suplementar a este artigo mostra o passo a passo matemático para a solução desta equação.[6]

Desse modo, a carteira de investimento é otimizada e os pesos ótimos $(\boldsymbol{\alpha_t})$ do modelo de múltiplos regimes em cada momento t são definidos pela equação (9):

$$\begin{aligned} \boldsymbol{\alpha_t} = {} & \frac{1}{\gamma}\left(\boldsymbol{D_{s,t}}\, \boldsymbol{\pi_t}\right)^T [(\boldsymbol{V\pi_t})(\boldsymbol{V\pi_t})^T]^{-1} \\ & + \frac{1}{\gamma}\textstyle\sum_{i=1}^{m}\left[A_i(\tau) + \; 2B_i(\tau)\, \pi_{i,t} + \; \sum_{j \neq i} C_{ij}(\tau)\, \pi_{i,t}\right] \boldsymbol{\sigma_{i,\pi}}\, (\boldsymbol{V\pi_t})^{-1}\,, \end{aligned} \tag{9}$$

onde $\boldsymbol{D_{s,t}}$ é a matriz $n \; x \; m$ que coleta os prêmios de risco de cada ativo de risco em suas linhas, com as colunas indicando tais prêmios de risco em cada um dos m regimes; $\pi_{i,t}$ é

[6] O documento suplementar ao artigo encontra-se disponível em: https://bibliotecadigital.fgv.br/ojs/index.php/rbfin/article/view/81210.

a probabilidade de se estar sob efeito do regime i no período t; $\boldsymbol{\pi_t}$ é um vetor coluna $m\ x\ 1$ que contém as m probabilidades de se estar, no período t, sob efeito de cada um dos m regimes; $\boldsymbol{V}$ é um vetor $1\ x\ m$ que contém como seus elementos m matrizes de covariância dos ativos condicionadas a cada um dos m regimes; A_i, B_i e C_{ij} são coeficientes dependentes do horizonte temporal que advêm da solução de otimização do sistema de equações diferenciais parciais oriundos da equação de Bellman; e, finalmente, $\boldsymbol{\sigma_{i,\pi}}$ é um vetor linha $1\ x\ n$ que coleta, para cada regime $i = 1, 2, \dots m$, as volatilidades parciais $\sigma_{ij,\pi}$ com $j = 1, 2, \dots n$: $\boldsymbol{\sigma_{i,\pi}} = [\ \sigma_{i1,\pi} \quad \sigma_{i2,\pi} \quad \dots \quad \sigma_{in,\pi}\ ]$. Os apêndices A e B fornecem mais detalhes matemáticos da configuração da economia bem como do processo de estimação dos parâmetros do modelo.

2.4.4 ALOCAÇÃO FORA DA AMOSTRA E REBALANCEAMENTO

Consideramos que nosso investidor aloca seus recursos baseado na informação corrente e aguarda até o período seguinte para refazer suas escolhas quando receberá novas informações do mercado, ou seja, quando conhecer os retornos daquele período. Cabe um comentário muito importante aqui: ao contrário de Lewin e Campani (2020), onde os autores estimam o modelo uma única vez com dados de toda a amostra, neste estudo reestimamos o modelo semestralmente e apenas com as informações disponíveis naquele momento.

Para definir a estratégia, *i.e.*, calcular os pesos na carteira de investimento, estimamos o modelo de 2009 até 2019, com a janela de informações fixada no início do período de análise e crescendo com o avançar do tempo. Esta janela temporal para estimação fora da amostra corresponde a utilização de cerca de 40% dos dados para calibragem (2001 até 2008) e 60% dos dados para gerar os resultados. O processo de atualização do modelo, isto é, a nova estimação dos parâmetros dos regimes, foi sempre realizado no primeiro fechamento semanal dos meses junho e dezembro. Por exemplo, ao estimarmos os parâmetros do modelo após o primeiro fechamento em dezembro de 2009, mantemos esses parâmetros até a última semana de maio de 2010, quando então repetimos o procedimento para o semestre seguinte com uma nova estimação realizada após o fechamento da primeira quarta-feira de junho de 2009. Isto torna os resultados aqui encontrados fora da amostra, ou seja, totalmente replicáveis por investidores no dia a dia – este é mais um ingrediente pioneiro deste artigo.

A respeito do período de rebalanceamento da carteira de investimento, quanto menor for este período, mais preciso ele será em relação ao modelo contínuo para o qual a solução analítica foi originalmente encontrada, porém maiores os custos de transação e de implementação da estratégia. Tais custos não foram incluídos, pois dependem (e muito) das características do investidor. Por consistência, nossos *benchmarks* igualmente não terão tais custos considerados.

2.4.5 CLASSES DE ATIVOS

Utilizamos a taxa do Certificado de Depósito Interfinanceiro (CDI) para denotar o investimento livre de risco (*cash*), pois além de representar um investimento amplamente difundido no Brasil, muito comumente é assim considerada em detrimento da taxa Selic, como apontam Andrino e Leal (2018). Não obstante, os resultados aqui encontrados não são sensíveis à mudança para a taxa Selic. Isto não causa espanto tendo em vista que ambas as taxas apresentaram comportamentos extremamente similares no período aqui analisado.

Os quatro ativos de risco foram representados por índices (ou taxas). Os dados foram extraídos do terminal Economática©, exceto a taxa livre de risco estadunidense, extraída do sistema *Board of Governors of the Federal Reserve*. Utilizamos fechamentos às quartas-feiras, compreendendo 849 observações de 24 de setembro de 2001 até 25 de dezembro de 2019. Nas datas em que não houve cotações, consideramos o fechamento anterior mais próximo. Optamos pela quarta-feira, ao invés da tradicional sexta-feira, para minimizar o efeito final de semana.

O Ibovespa foi adotado como a renda variável por ser um índice da B3 amplamente utilizado para representar o mercado brasileiro de ações, segundo Oliveira e Pereira (2018). Como investimento alternativo negativamente correlacionado à renda variável, denotamos como *carry-trade* a taxa do US Treasury Bill de 3 meses (T-Bill) convertido em reais pela taxa PTAX, computada pelo Banco Central do Brasil. Por último, segmentamos a classe de renda fixa entre IMA-B 5 (curto prazo) e IMA-B 5+ (longo prazo). Estes dois índices são calculados pela Anbima e compostos por títulos do Governo brasileiro (NTN-B) pós-fixados lastreados pelo índice inflacionário IPCA. Ressalte-se que o IMA-B 5, por possuir títulos com até 5 anos de vencimento, apresenta retorno e volatilidade muito semelhantes e alta correlação com o IRF-M (também calculado pela Anbima), que por sua vez, representa uma carteira com títulos pré-fixados (LTN e NTN-F). Por conta disso, o IRF-M foi desconsiderado como classe de ativos: ao incluí-lo, o modelo não era estimado de forma tão robusta e oscilações injustificáveis ocorriam por conta da alta correlação com o IMA-B 5.

Reconhecemos certo sobrepeso das classes de renda fixa em relação à renda variável na carteira selecionada: de um lado há dois ativos referentes a títulos do governo brasileiro e o CDI, e do outro, um ativo referente às ações brasileiras. Entretanto, há de se ponderar em separado a classificação atribuída ao *carry-trade* convertido em reais. Ao formar esta classe, verificamos que a PTAX apresenta impacto muito mais significativo do que o retorno da T-Bill. O objetivo de criá-la foi representar a alternativa do investidor brasileiro, que compra dólares para se proteger – e quando o faz, indiretamente aplica em algum investimento. Selecionamos a taxa livre de risco dos EUA para resumir o risco da classe apenas ao câmbio. Apesar da PTAX/T-Bill não ser formalmente uma renda variável como ações, a classe encontra forte correlação negativa com o IBOV. Portanto, em relação à composição da carteira, poderíamos separar 3 ativos ligados à renda fixa e 2 ativos ligados à renda variável.

Não obstante, os aspectos que motivaram tal seleção de ativos foram: a importância do mercado de renda fixa brasileiro e a pesquisa realizada para se chegar nesta carteira. Com o objetivo de realizar *backtests* fora da amostra, buscamos uma carteira comum a todos os anos da amostra, sem alterar os ativos a cada ano. O mercado de renda fixa nacional, embora a taxa básica de juros tenha sofrido cortes desde meados de 2016-17, possui relevante importância frente à renda variável no Brasil. Como *proxy*, tomemos os dados sobre a indústria de fundos de investimento no Brasil (Anbima, 2020). O histórico mostra que o patrimônio líquido dos fundos de renda fixa deteve participação significativamente maior do que as demais classes de investimentos em todos os anos entre dez/2009 e dez/2019. Por fim, o outro aspecto determinante à seleção das classes de ativos adveio da pesquisa realizada em 2019 junto a fundos de pensão, que alertaram possuir (até aquele momento) restrições quanto à exposição a ativos de renda variável no exterior e a ativos de crédito privado.

2.4.6 ABORDAGENS UTILIZADAS NA AVALIAÇÃO DA PERFORMANCE

Para avaliar a performance da carteira sugerida pelo modelo de múltiplos regimes (modelo CGL), observamos os resultados dos índices representativos das classes de ativos (CDI, IBOV, PTAX+T-Bill, IMA-B 5+ e IMA-B 5); e criamos duas carteiras de referência. A primeira utiliza a função estocástica diferencial, mas enxerga apenas um regime na economia (regime único); e a segunda representa a carteira igualmente ponderada, ou seja, sempre rebalanceada uniformemente para 20% em cada classe (denotada como $1/n$ ou pesos iguais).

A comparação do modelo CGL com a carteira de regime único, também classificada como carteira míope por não enxergar os múltiplos regimes, é relevante na medida em que ambos os modelos utilizam a mesma função utilidade. Portanto, esta comparação permite avaliar a importância do modelo de múltiplos regimes na alocação de ativos.

Já a carteira igualmente ponderada denota uma média linear dos retornos dos ativos, o que, em última instância, representa um investidor que não possui um modelo de alocação superior à própria média. DeMiguel, Garlappi e Uppal (2009) recomendam a utilização desta carteira como *benchmark* para a performance de portfólios ao invés de outros modelos que otimizam a alocação para eliminar da comparação a possibilidade de erros de estimação. Na mesma linha, Kessler e Scherer (2009) afirmam que uma carteira de pesos iguais permite avaliar de forma mais clara as diferenças entre performances comparadas ao invés de um portfólio de média-variância, por exemplo, uma vez que este demandaria a estimação de uma matriz de covariâncias e eventuais erros de estimação prejudicariam a comparação com outros métodos. No Brasil, Santiago e Leal (2015) mostram a relevância da carteira de pesos iguais através de sua comparação empírica com 52 fundos de ações: apenas 2 fundos superaram os retornos da carteira $1/n$. Não obstante, Andrino e Leal (2018) sugerem que portfólios de renda fixa com pesos iguais podem superar fundos balanceados. Desse modo, a literatura sugere a comparação com a carteira de pesos iguais para se avaliar a importância de um modelo na gestão ativa de portfólios. No caso, a gestão ativa se refere às estratégias de portfólio que visam maximizar retornos permitindo rebalanceamentos dinâmicos na alocação dos ativos da carteira.

2.5 RESULTADOS, ANÁLISES E DISCUSSÕES

2.5.1 MODELO DE MÚLTIPLOS REGIMES

Tabela 1 – Parâmetros estimados para os modelos de múltiplos regimes

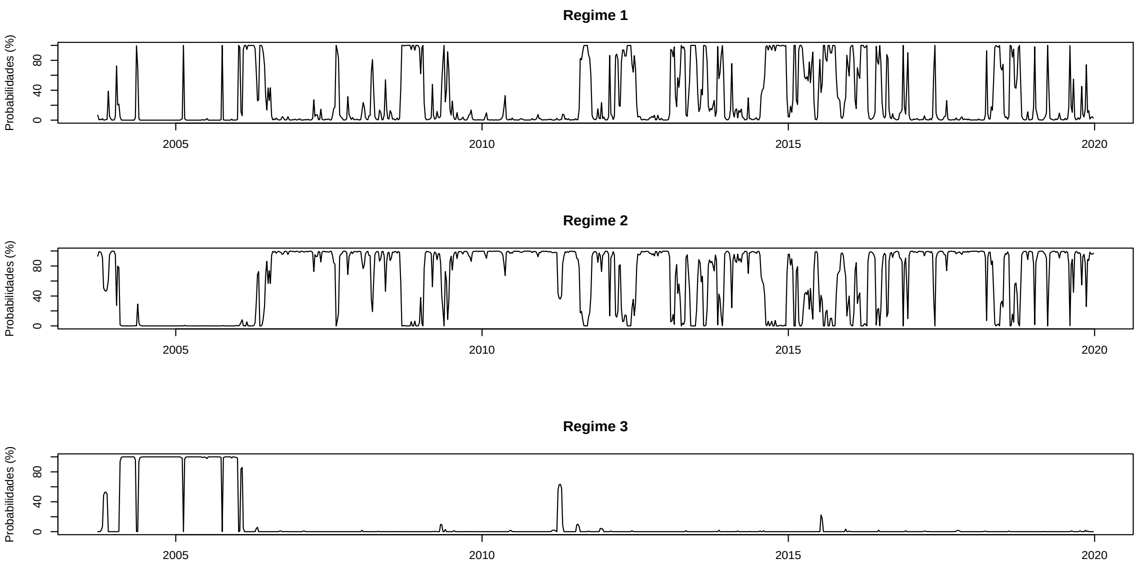
	IBOV	PTAX+T-Bill	IMA-B 5+	IMA-B 5
Painel A: Modelo Míope (Enxerga um único regime na economia)				
Retorno Médio dos Prêmios de Risco	4%	-8%	6%	2%
Matriz de Correlação e Volatilidade				
IBOV	24%			
PTAX+T-Bill	-50%	15%		
IMA-B 5+	36%	-41%	10%	
IMA-B 5	29%	-31%	77%	3%
Painel B: Modelo de Múltiplos Regimes				
Retorno Médio dos Prêmios de Risco				
Regime 1	-33%	36%	-6%	-1%
Regime 2	21%	-17%	12%	4%
Regime 3	16%	-28%	-2%	-2%
Matriz de Correlação e Volatilidade				
Regime 1				
IBOV	33%			
PTAX+T-Bill	-54%	24%		
IMA-B 5+	45%	-48%	16%	
IMA-B 5	39%	-37%	78%	4%
Regime 2				
IBOV	20%			
PTAX+T-Bill	-47%	10%		
IMA-B 5+	25%	-28%	6%	
IMA-B 5	19%	-20%	75%	2%
Regime 3				
IBOV	22%			
PTAX+T-Bill	-30%	11%		
IMA-B 5+	-7%	6%	1%	
IMA-B 5	6%	-8%	52%	1%
Probabilidades de Transição Semanal	**Regime 1**	**Regime 2**	**Regime 3**	**Duração**
Regime 1	75%	23%	2%	4 semanas
Regime 2	9%	90%	1%	10 semanas
Regime 3	6%	1%	93%	15 semanas

Nota. A tabela mostra as classes de ativos de risco. Os retornos estão configurados como prêmios de risco em relação ao CDI (classe *cash*). As matrizes de correlação apresentam as volatilidades em suas diagonais principais. Retornos e volatilidades originam-se dos dados semanais e foram anualizados para apresentação. Os parâmetros foram estimados em 4 de dezembro de 2019.

A tabela 1 apresenta os parâmetros dos regimes identificados na última estimação do modelo, em 04.12.2019. Observamos que o regime 1 é o mais volátil e apesar de geralmente isto indicar um movimento pessimista da economia, eventualmente compreende situações de forte volatilidade positiva, o que é comumente conhecido no mercado como um “rally”. O regime 2 engloba a maioria das flutuações do mercado, enquanto o regime 3 denota a alta da renda variável junto com o CDI rendendo acima das classes de juros.

A figura 1 ilustra a probabilidade histórica dos três regimes identificados. Fica nítido que o terceiro regime mostra-se “adormecido” na maior parte do tempo. Isto ocorre pois o modelo atribui ao terceiro regime o estado da economia observado em meados de 2005, quando o Brasil vivia um super ciclo nas *commodities*, de forma que posteriormente tal regime ficou subutilizado, operando o modelo mormente com apenas dois regimes.

Figura 1 – Probabilidade de ocorrência dos regimes

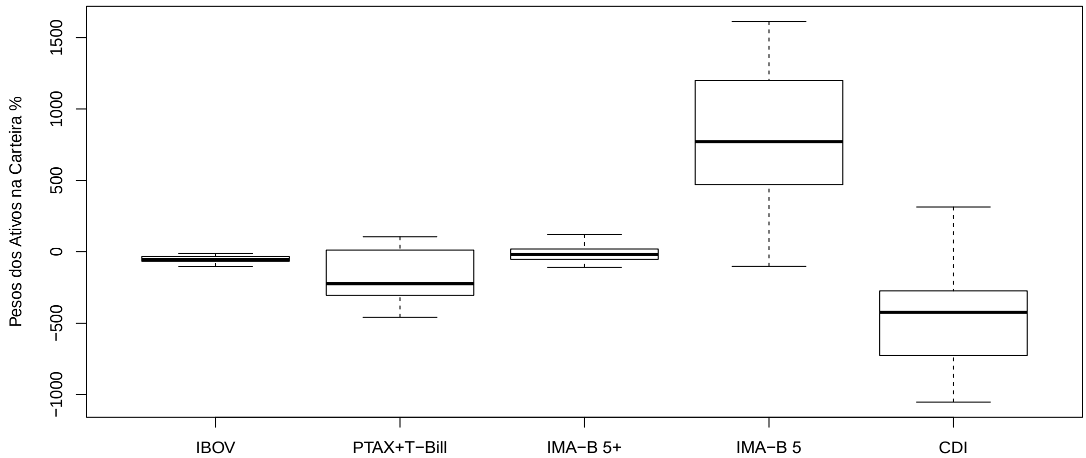


Nota: A figura ilustra as probabilidades históricas semanais de ocorrência de cada regime. Pelo caráter informacional (e não de investimento), trata-se da probabilidade suavizada, isto é, que reúne todas as informações do período completo até dezembro de 2019. Os dados correspondem à última estimação do modelo, realizada em 04.12.2019.

2.5.2 MODELO DE ALOCAÇÃO

A figura 2 apresenta um aspecto relevante da estratégia de alocação que é a forte alavancagem indicada pelo modelo CGL. A literatura aponta que modelos similares também apresentam esse resultado em função do maior conhecimento em relação aos estados econômicos (Sangvinatsos e Wachter, 2005). Isto faz com que o investidor tome posições bastante alavancadas. Não obstante, Lewin e Campani (2020) mostram que um menor nível de alavancagem não altera a formatação da carteira CGL, porém apenas a magnitude de seus retornos. Desse modo, utilizamos um maior nível alavancagem para apresentar o potencial do modelo. Consideramos que as posições vendidas em títulos de renda fixa ou *cash*, por exemplo, podem ser replicadas com posições equivalentes no mercado brasileiro de derivativos de juros.

Figura 2 – Distribuição dos pesos na carteira CGL



Nota: A figura ilustra a distribuição dos pesos na carteira formada pelo modelo CGL com base na estimação fora da amostra durante a janela de observações de 02.12.2009 até 04.12.2019. *Outliers* foram suprimidos da imagem.

A figura 2 mostra que o modelo frequentemente indica forte posição comprada na classe de juros de curto prazo (IMA-B 5), que por sua vez possui alta correlação com juros prefixados. Em contrapartida, as maiores posições vendidas a descoberto são no ativo livre de risco (CDI) e depois no ativo de juros estadunidenses convertido em reais (PTAX+T-Bill): isso indica que o maior conhecimento a respeito dos regimes faz com que o investidor tome emprestado no Brasil e nos EUA para alavancar posições nos mercados de renda fixa e variável brasileiros.

Por outro lado, o modelo opera com volume nitidamente menor tanto na renda variável (IBOV) quanto nos juros de longo prazo (IMA-B 5+). No caso da renda variável, a maior proporção de renda fixa na carteira "força" o modelo na direção de três regimes. Isto é encontrado na literatura em Guidolin (2011) que aponta que a estimação de "apenas" três regimes implica em menor coerência na diversificação com outras classes de investimento. Com os regimes capturando relativamente menos nuances da renda variável, o modelo se torna menos flexível, implicando em menor volume na alocação desta classe. No caso da renda fixa de longo prazo, como o modelo encontra rentabilidade superior na operação fortemente alavancada em renda fixa, a ponta comprada da operação é direcionada ao ativo correlacionado com melhor relação de risco e retorno. Assim, (i) embora o IMA-B 5+ apresente maior retorno esperado que o IMA-B 5; e (ii) porque há alta correlação entre ambos em todos os regimes (mas principalmente nos dois regimes mais comuns: 1 e 2) – a menor volatilidade induz o maior volume da carteira ser direcionado à alocação em renda fixa de curto prazo.

Através da tabela 2 é possível comparar as performances ano a ano e acumuladas no período completo entre 2009 e 2019. Como salientado, a estimação foi realizada totalmente fora da amostra para permitir a avaliação dos resultados de forma realista e passível de aplicações práticas.

Tabela 2 – Performance comparativa

Ano		**CDI**	**IBOV**	**PTAX+T-Bill**	**IMA-B 5+**	**IMA-B 5**	**Modelo CGL**	**Regime Único**	**Pesos Iguais**
2019	Retorno	6,1%	20,7%	10,3%	29,5%	13,3%	98,3%	29,5%	16,2%
	Vol	0,1%	17,7%	13,1%	8,7%	2,3%	34,6%	12,5%	4,5%
	IS	-	0,8	0,3	2,7	3,2	2,7	1,9	2,3
	Turnover	-	-	-	-	-	18,6%	0,3%	0,5%
2018	Retorno	6,4%	22,8%	20,2%	13,5%	9,2%	-24,8%	-2,3%	14,8%
	Vol	0,1%	19,4%	13,1%	8,5%	2,8%	37,0%	16,6%	5,0%
	IS	-	0,8	1,1	0,8	1,0	-0,8	-0,5	1,7
	Turnover	-	-	-	-	-	15,7%	0,3%	0,5%
2017	Retorno	10,5%	17,4%	-5,4%	16,5%	13,2%	83,6%	33,3%	10,6%
	Vol	0,3%	18,9%	10,2%	9,4%	2,6%	44,6%	12,7%	4,5%
	IS	-	0,4	-1,6	0,6	1,0	1,6	1,8	0,0
	Turnover	-	-	-	-	-	17,4%	0,5%	0,5%
2016	Retorno	14,2%	32,1%	-9,9%	25,6%	15,5%	-26,1%	25,1%	15,7%
	Vol	0,2%	26,0%	17,9%	10,6%	2,7%	53,5%	18,3%	5,3%
	IS	-	0,7	-1,3	1,1	0,5	-0,8	0,6	0,3
	Turnover	-	-	-	-	-	15,6%	0,5%	0,7%
2015	Retorno	12,9%	-14,9%	50,1%	2,6%	13,8%	-26,7%	-5,4%	12,0%
	Vol	0,2%	22,0%	19,4%	11,9%	3,0%	41,1%	24,5%	5,6%
	IS	-	-1,3	1,9	-0,9	0,3	-1,0	-0,7	-0,2
	Turnover	-	-	-	-	-	23,6%	0,8%	0,7%
2014	Retorno	10,6%	6,2%	8,9%	21,0%	13,1%	70,0%	13,9%	12,6%
	Vol	0,1%	23,9%	11,8%	13,9%	2,6%	40,2%	20,5%	6,2%
	IS	-	-0,2	-0,1	0,7	1,0	1,5	0,2	0,3
	Turnover	-	-	-	-	-	13,4%	0,5%	0,6%
2013	Retorno	7,8%	-8,3%	10,3%	-15,0%	3,2%	-59,9%	-18,5%	-0,3%
	Vol	0,2%	21,1%	11,4%	14,9%	4,3%	54,9%	30,2%	6,0%
	IS	-	-0,8	0,2	-1,5	-1,1	-1,2	-0,9	-1,3
	Turnover	-	-	-	-	-	25,1%	0,7%	0,6%
2012	Retorno	8,7%	-0,6%	15,5%	31,3%	15,2%	98,3%	33,5%	14,3%
	Vol	0,2%	23,1%	11,2%	8,8%	2,5%	33,9%	21,0%	4,3%
	IS	-	-0,4	0,6	2,6	2,5	2,6	1,2	1,3
	Turnover	-	-	-	-	-	21,1%	0,9%	0,5%
2011	Retorno	11,8%	-18,3%	5,0%	19,3%	17,1%	128,7%	46,2%	6,6%
	Vol	0,2%	21,3%	12,7%	6,5%	2,1%	30,0%	19,6%	4,3%
	IS	-	-1,4	-0,5	1,2	2,5	3,9	1,8	-1,2
	Turnover	-	-	-	-	-	17,6%	1,5%	0,5%
2010	Retorno	9,5%	2,5%	-0,1%	19,3%	12,7%	54,7%	30,0%	9,1%
	Vol	0,2%	20,8%	9,7%	5,2%	1,2%	20,9%	10,6%	3,3%
	IS	-	-0,3	-1,0	1,9	2,8	2,2	1,9	-0,1
	Turnover	-	-	-	-	-	9,3%	0,8%	0,4%
Acum.	Retorno	9,8%	4,7%	9,5%	15,5%	12,6%	21,6%	16,7%	11,1%
	Vol	0,4%	21,5%	13,4%	10,3%	2,7%	40,6%	19,5%	5,0%
	IS	-	-0,2	0,0	0,6	1,0	0,3	0,4	0,2
	Turnover	-	-	-	-	-	17,7%	0,7%	0,5%

Nota. A tabela apresenta as performances das classes de ativos, da carteira CGL e de duas carteiras de referência para comparação. Os períodos anuais compreendem os retornos entre as estimações realizadas no início de cada dezembro. Já o período acumulado (*Acum.*) corresponde aos retornos obtidos entre as estimações de dezembro 2009 e dezembro 2019, e está apresentado anualizado. Os dados estão apresentados em retornos absolutos (e não em prêmio de risco). Retorno indica os retornos totais; vol, a volatilidade ao ano (desvio padrão anualizado); IS, o índice de Sharpe em base anual; enquanto *turnover* denota a rotação média da carteira por semana.

Na tabela 2 pode ser verificado um outro aspecto particular às características dos retornos em tela que leva à maior exposição à IMA-B5 e CDI em detrimento dos demais ativos. Tomando o ano de 2019 como exemplo, a tabela 2 mostra que IBOV e IMA-B5+ foram os ativos mais rentáveis no ano, entretanto a carteira CGL apresenta retorno significativamente superior a esses ativos com baixa exposição a eles (como vimos na figura 2). Ao mesmo tempo, as probabilidades históricas (dados da figura 1) mostram que dentre as 52 semanas de 2019, o regime 2 vigorou como o mais provável em 47 delas. A tabela 1 indica que sob o regime 2 as volatilidades de IBOV e IMA-B5+ respectivamente seriam 10 e 5 vezes acima da volatilidade do IMA-B5. Enquanto o retorno esperado de ambos, respectivamente, seria "apenas" 5 e 3 vezes superior ao IMA-B5. Isso justifica a preferência do modelo a operar o IMA-B5 ao invés de IBOV e IMA-B5+ mesmo sob um cenário de alta rentabilidade esperada. Em outras palavras, além do fato de que ao trabalhar sob uma função dinâmica e não sob uma função estática como a função quadrática (média-variância), o modelo consegue realizar *hedging demands* – a alocação proposta pelo modelo CGL reflete o *trade-off* retorno *versus* volatilidade à exemplo de um investidor que analisa o índice de Sharpe e não apenas retornos em sua decisão.

2.5.3 PERFORMANCE FORA DA AMOSTRA

Na análise ano-a-ano, a tabela 2 mostra que os retornos do modelo CGL superam todos os *benchmarks* em 6 dos 10 anos observados na amostra. A magnitude dos retornos CGL, tanto para cima quanto para baixo, é visivelmente maior que de seus pares, e isto decorre de uma alavancagem elevada – o que também se traduz por significativa volatilidade. Note-se que o índice Sharpe da carteira CGL foi superior às duas carteiras de referência em 5 dos 10 anos analisados, ao passo que venceu todas as classes de ativos (especialmente o IMA-B 5) em apenas 4 anos, o que se explica pela baixíssima volatilidade apresentada pelo IMA-B 5.

A tabela indica que os resultados do modelo CGL são obtidos a partir de um *turnover* médio de 18% por semana (girando historicamente entre 9% e 25%) sobre o volume da carteira alavancada, bastante acima das carteiras pares – isso se dá porque a carteira CGL procura antecipar a mudança de regimes, que pode acarretar mudanças bruscas na carteira. Acompanhamos Chen, Sanger e Slovin (2013) para determinar o cálculo do percentual de *turnover* como sendo o valor do montante negociado sobre o valor total da carteira, considerando, claro, a alavancagem. Segundo os autores, o valor do montante negociado pode ser obtido seja através do valor dos novos ativos comprados, seja através do valor absoluto dos ativos vendidos – uma vez que o consumo é desconsiderado. Entretanto, frisamos, que o montante negociado não corresponde à compra mais a venda semanal.

Os resultados acumulados durante toda a janela de estimações são apresentados na última parte da tabela 2. Analisando os retornos obtidos, verificamos que entre os ativos destacam-se o IMA-B 5+ (15,5% a.a.) e IMA-B 5 (12,6% a.a.). Enquanto isso, a performance do modelo CGL supera-os, bem como supera as carteiras regime único (16,7% a.a.) e pesos iguais (11,1% a.a.), rendendo 21,6% ao ano no período! Esse resultado sugere que não apenas que a observância dos regimes é relevante na formação da carteira, mas também que o modelo se destaca da carteira ingênua $1/n$, ratificando a importância do modelo CGL para estratégias que utilizam a gestão ativa do portfólio.

Para alcançar tal magnitude de retorno, o modelo CGL opera deveras alavancado. Utilizando o índice de Sharpe como métrica de retorno *vis-a-vis* o risco tomado, a volatilidade superior do modelo CGL é destacada através do índice de Sharpe menor do que nas classes de renda fixa e ligeiramente menor do que na carteia míope. Por outro lado, apesar de sua volatilidade, o modelo CGL supera os demais ativos e carteiras neste indicador de performance.

Para avaliar a robustez dos retornos do modelo CGL, observamos as séries semanais no período completo de análise (2010 a 2019). Realizamos dois testes de hipótese com a finalidade de avaliar se a média dos retornos da carteira CGL excederia a média dos retornos de cada um dos *benchmarks* (classes de ativos e carteiras de referência) com significância estatística.

Primeiro realizamos o teste não paramétrico Wilcoxon *Rank Sum* (também conhecido como teste Mann-Whitney) acompanhando Copeland e Friedman (1991) e Bergmann, Saviola, De Angelo, Contani e Silva (2018). A tabela 3 apresenta os p-valores para a hipótese nula de que a média dos retornos da carteira CGL não seria maior que dos *benchmarks*. Note-se que todos os resultados ficaram abaixo de 5%, indicando que no primeiro teste de robustez, a carteira CGL apresentou retornos acima dos *benchmarks* com significância estatística.

Reiczigel, Zakariás e Rózsa (2005) afirmam que o teste Wilcoxon é amplamente utilizado para comparar séries com distribuição não-normal. Apesar disso, questionam a existência de premissas possivelmente falhas neste teste. Portanto, como sugerido pelos mesmos autores, adicionalmente apresentamos o teste de hipótese por *bootstrap* de Efron e Tibshirani (1993). A vantagem deste segundo teste é que gerando uma quantidade suficientemente elevada de novas amostras, ele permite o relaxamento da premissa quanto ao tipo de distribuição uma vez que sob a lei dos grandes números, é possível replicar as caraterísticas da distribuição original através do processo de reamostragem aleatória. A tabela 3 também apresenta os p-valores deste teste, onde os resultados indicam significância estatística na ordem de 8%, exceto na comparação com PTAX-T-Bill e regime único cuja significância foi aproximadamente 12%.

Ainda que os resultados do segundo teste não tenham sido tão contundentes quanto no teste anterior, eles sugerem direção semelhante. Por isso, há necessidade de avaliação da robustez através de testes com premissas distintas. A análise conjunta fortalece a conclusão de que a performance da carteira CGL (em termos de retorno médio) supera estatisticamente as médias dos demais investimentos analisados.

Tabela 3 – Comparação entre modelo CGL e *benchmarks*

Teste	**CDI**	**IBOV**	**PTAX+ T-Bill**	**IMA-B 5+**	**IMA-B 5**	**Regime Único**	**Pesos Iguais**
Wilcoxon	0,0001	0,0013	0,0002	0,0011	0,0002	0,0218	0,0001
Bootstrap	0,0855	0,0819	0,1138	0,0624	0,0772	0,1203	0,0859

Nota. A tabela apresenta os p-valores dos testes estatísticos, ambos realizados sob a hipótese nula de que a média dos retornos da carteira CGL não seria maior que a média dos *benchmarks*. O período testado corresponde aos retornos semanais entre 2 de dezembro 2009 e 4 de dezembro 2019. Wilcoxon corresponde ao teste unicaudal Wilcoxon *Rank Sum and Signed Rank* gerado a partir da função wilcox.test residente no programa estatístico R (R Core Team, 2019). Tal função acompanha os trabalhos de Hollander e Wolfe (1999) e Bauer (1972). *Bootstrap* refere-se ao teste de Efron e Tibshirani (1993). O resultado do *bootstrap* foi calculado com 1 milhão de amostras.

As performances apresentadas nesta seção sugerem que a carteira CGL faz sentido e possua um nicho de mercado de interesse. Tal nicho engloba investidores com objetivos de longo prazo e que se sujeitem a volatilidades altas. Como a performance resulta de retornos significativamente alavancados, a estratégia levaria investidores de curto prazo – como, por exemplo, investimentos com horizonte de apenas um ano – a uma altíssima exposição de risco: para investidores com horizontes mais longos, tal exposição dissipa-se (ao menos, parcialmente). Ademais, a estratégia requer uma gestão bastante ativa (i.e., com rebalanceamentos semanais), indicando a necessidade de gestão profissional (i.e., com rebalanceamentos eficientes) e escala para operar sob custos transacionais controlados mesmo diante de elevado *turnover*.

2.6 CONCLUSÃO

Esta pesquisa inova na aplicação do modelo CGL com quatro classes de ativos de risco, e principalmente por ser a primeira avaliação da performance de uma alocação com estimações totalmente fora da amostra para um modelo de múltiplos regimes sob a função estocástica diferencial. Isto representa um passo importantíssimo neste campo da literatura internacional pois os resultados aqui obtidos são realistas e poderiam ser obtidos por gestores no mercado.

Mostramos que a performance média alcançada pelo modelo CGL superou todos os *benchmarks* (classes de ativos e carteiras de referência) em termos de retornos semanais médios acumulados entre dez/2009 e dez/2019, sempre com significância estatística. Nesse período, a nossa estratégia alcançou um retorno médio igual a 21,6% ao ano contra, por exemplo, 9,8% a.a. do CDI e 4,7% a.a. do Ibovespa. E na análise separada de cada ano, em 6 dos 10 anos em tela, o modelo CGL apresentou média de retorno semanal superior a qualquer dos *benchmarks*. Tais performances corroboram os resultados anteriores que no entanto haviam sido obtidos dentro da amostra (Campani, Garcia e Lewin, 2020; Lewin e Campani, 2020).

Nossos resultados também indicam que ao superar estatisticamente os retornos médios da carteira igualmente ponderada, o modelo CGL se destaca na gestão ativa de portfólios. Da mesma forma, superando a carteira de regime único, torna-se evidente a importância da adoção do modelo de múltiplos regimes em estratégias de alocação de ativos.

Ainda assim, a estratégia proveniente do modelo CGL deve ser implementada por gestores profissionais, pois há necessidade de alto *turnover* semanal: os rebalanceamentos precisam ser feitos eficientemente e com custos transacionais controlados. Além disso, ela não é indicada para investidores de curto prazo, pois o risco pode ser alto demais para estes investidores: o dinamismo da estratégia ao trocar os regimes econômicos faz a mesma ter volatilidade acima dos índices tradicionais de mercado.

Os resultados encontrados nesta pesquisa motivam a extensão em novas aplicações do modelo CGL nos mercados brasileiro e internacional. Como próximos passos, avaliamos aplicar o modelo CGL separadamente à cada classe de ativo para permitir, por exemplo, ao investidor definir a própria calibragem entre renda fixa e renda variável durante o período de observação. Em outra frente, o fato de não incluirmos nos resultados custos de transação e de liquidez do mercado, aliados às elevadas posições vendidas, representam potenciais limitações à aplicação da estratégia e isto também deverá ser analisado em pesquisa futura. Não obstante, os resultados encontrados soam bastante promissores e abrem caminho para a utilização de modelos de múltiplos regimes na alocação de carteiras de investimento no Brasil.

2.7 REFERÊNCIAS

Anbima (2020). Consolidado Histórico de Fundos de Investimento – Patrimônio Líquido por Classe Anbima. Disponível em https://www.anbima.com.br/pt_br/informar/estatisticas/fundos-de-investimento/fi-consolidado-historico.htm (acesso em 28/5/2020).

Andrino, Daniel D. & Leal, Ricardo P.C. (2018). Equally Weighted Strategic Allocation and Balanced Funds in Brazil. *Latin American Business Review*, *19*(1), 23-53. https://doi.org/10.1080/10978526.2018.1450150

Ang, Andrew & Bekaert, Geert (2002). International Asset Allocation with Regime Shifts. *Review of Financial Studies*, *15*(4), 1137-1187. https://doi.org/10.1093/rfs/15.4.1137

Ang, Andrew & Bekaert, Geert (2004). How Regimes Affect Asset Allocation. *Financial Analysts Journal*, *60*(2), 86-99. https://doi.org/10.2469/faj.v60.n2.2612

Ang, Andrew & Timmermann, Allan (2012). Regime Changes and Financial Markets. *Annual Review of Financial Economics*, *4*(1), 313-337. https://doi.org/10.1146/annurev-financial-110311-101808

Bae, Geum; Kim, Woo & Mulvey, John (2014). Dynamic asset allocation for varied financial markets under regime switching framework. *European Journal of Operational Research*, *234*(2), 450-468. https://doi.org/10.1016/j.ejor.2013.03.032

Bauer, David (1972). Constructing confidence sets using rank statistics. *Journal of the American Statistical Association*, *67*(339), 687–690. https://doi.org/10.1080/01621459.1972.10481279

Bergmann, Daniel; Savoia, José; De Angelo, Claudio; Contani, Eduardo & Silva, Fabiana (2018). Portfolio management with tail dependence. *Applied Economics*, *50*(51), 5510–5520. https://doi.org/10.1080/00036846.2018.1487000

Copeland, Thomas & Friedman, Daniel (1991). Partial Revelation of Information in Experimental Asset Markets. *The Journal of Finance*, *46*(1), 265–295. https://doi.org/10.1111/j.1540-6261.1991.tb03752.x

Campani, Carlos Heitor & Garcia, René (2019). Approximate Analytical Solutions for Consumption/Investment Problems under Recursive Utility and Finite Horizon. *North American Journal of Economics and Finance*, *48*, 364-384. https://doi.org/10.1016/j.najef.2019.03.005

Campani, Carlos Heitor; Garcia, René; & Lewin, Marcelo (2021). Optimal Portfolio Strategies in the Presence of Regimes in Asset Returns. *Journal of Banking and Finance*, *123*(106030), 1–17. https://doi.org/10.1016/j.jbankfin.2020.106030

Case, Brad; Guidolin, Massimo & Yildirim, Yildiray (2014). Markov switching dynamics in REIT returns: Univariate and multivariate evidence on forecasting performance. *Real Estate Economics*, *42*(2), 279-342. https://doi.org/10.1111/1540-6229.12025

Chen, Fan; Sanger, Gary & Slovin, Myron (2013). Asset sales in the mutual fund industry: Who gains? *Journal of Banking and Finance*, *37*(12), 4834–4849. https://doi.org/10.1016/j.jbankfin.2013.08.019

DeMiguel, Victor; Garlappi, Lorenzo; & Uppal, Raman (2009). Optimal versus naive diversification: How inefficient is the 1/N portfolio strategy? *Review of Financial Studies*, *22*(5), 1915–1953. https://doi.org/10.1093/rfs/hhm075

Duffie, Darrell & Epstein, Larry (1992). Stochastic Differential Utility. *Econometrica*, *60*(2), 353-394. https://www.jstor.org/stable/2951600

Efron, Bradley & Tibshirani, Robert (1993). *An Introduction to the Bootstrap*. Chapman & Hall, New York.

Graflund, Andreas & Nilsson, Birger (2003). Dynamic Portfolio Selection: The Relevance of Switching Regimes and Investment Horizon. *European Financial Management*, *9*(2), 179-200. https://doi.org/10.1111/1468-036X.00215

Guidolin, Massimo (2011). Markov switching models in empirical finance. *Advances in Econometrics*, *27*(2), 1-86. https://doi.org/10.1108/S0731-9053(2011)000027B004

Guidolin, Massimo & Hyde, Stuart (2012). Can VAR Models Capture Regime Shifts in Asset Returns? A Long-Horizon Strategic Asset Allocation Perspective. *Journal of Banking and Finance*, *36*(3), 695-716. https://doi.org/10.1016/j.jbankfin.2011.10.011

Guidolin, Massimo & Timmermann, Allan (2007). Asset Allocation under Multivariate Regime Switching. *Journal of Economic Dynamics and Control*, *31*(11), 3503-3544. https://doi.org/10.1016/j.jedc.2006.12.004

Guidolin, Massimo, & Ono, Sadayuki (2006). Are the dynamic linkages between the macroeconomy and asset prices time-varying? *Journal of Economics and Business*, *58*(5–6), 480–518. https://doi.org/10.1016/j.jeconbus.2006.06.009

Hamilton, James (1989). A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle. *Econometrica*, *57*(2), 357-384. https://www.jstor.org/stable/1912559

Hollander, Myles & Wolfe, Douglas (1999). *Nonparametric Statistical Methods*. John Wiley & Sons, New York, 2nd ed,. www.jstor.org/stable/2063815

Kessler, Stephan & Scherer, Bernd (2009). Varying risk premia in international bond markets. *Journal of Banking and Finance*, *33*(8), 1361–1375. https://doi.org/10.1016/j.jbankfin.2009.02.007

Lewin, Marcelo & Campani, Carlos Heitor (2020). Portfolio Management Under Multiple Regimes: Strategies that outperform the Market. *RAC – Revista de Administração Contemporânea*, 24(4), 300-3016. https://doi.org/10.1590/1982-7849rac2020190161

Munk, Claus (2013). *Financial Asset Pricing Theory*. Oxford University Press, London, 6th ed. https://doi.org/10.1093/acprof:oso/9780199585496.001.0001

Oliveira, André & Pereira, Pedro (2018). Uncertainty Times for Portfolio Selection at Financial Market [Working Paper nº 473]. *CEQEF – FGV Sao Paulo School of Economics*, São Paulo, SP. http://dx.doi.org/10.2139/ssrn.3143092

R Core Team (2019). "R: A Language and Environment for Statistical Computing", *R Foundation for Statistical Computing*. https://www.R-project.org/

Reiczigel, Jenö; Zakariás, Ildikó & Rózsa, Lajos (2005). A bootstrap test of stochastic equality of two populations. *American Statistician*, *59*(2), 156–161. https://doi.org/10.1198/000313005X23526

Sangvinatsos, Antonios & Jessica Wachter (2005). Does the Failure of the Expectations Hypothesis Matter for Long-Term Investors? *Journal of Finance 60*(1), 179-230. https://doi.org/10.1111/j.1540-6261.2005.00728.x

Santiago, Diogo C. & Leal, Ricardo P.C (2015) Carteiras igualmente ponderadas com poucas ações e o pequeno investidor. *RAC – Revista de Administração Contemporânea*, *19*(5), 544-564, http://dx.doi.org/10.1590/1982-7849rac20151794

Tu, Jun (2010). Is Regime Switching in Stock Returns Important in Portfolio Decisions? *Management Science*, *56*(7), 1198-1215. https://doi.org/10.1287/mnsc.1100.1181

2.8 APÊNDICE

A. Detalhamento da Configuração da Economia

Como os estados da economia não são observáveis, assumimos que o investidor enxerga os prêmios de risco como um processo de média ponderada pelas probabilidades de ocorrência de cada regime em t:

$$\boldsymbol{d\hat{r}_t} = \boldsymbol{D_{s,t}}\boldsymbol{\pi_t}dt + (\boldsymbol{V}\boldsymbol{\pi_t})\, \boldsymbol{dZ_t}, \tag{10}$$

onde $\boldsymbol{d\hat{r}_t}$ é o vetor coluna com todos os n processos dos prêmios de risco; $\boldsymbol{D_{s,t}}$ é o elemento configurado na equação (4) como a matriz ($n\ x\ m$) contendo os *drifts* dos ativos por regime; e $\boldsymbol{\pi_t}$ é um vetor coluna de dimensão é $m\ x\ 1$ que arquiva as probabilidades (filtradas) de estar em cada possível estado da economia no momento t condicionadas às informações disponíveis até aquele determinado momento t, sendo:

$$\boldsymbol{\pi_t} = [\, \pi_{1,t} \quad \pi_{2,t} \quad \dots \quad \pi_{m,t}\,]^T, \tag{11}$$

e $\boldsymbol{V}$ é um vetor linha de matrizes com dimensão $1\ x\ m$ que armazena as matrizes $\boldsymbol{\sigma_{s,i}}$ que, como detalhado na equação (3), são constantes e especificadas para cada regime ($i \in R$, sendo $R = \{1, 2, \dots, m\}$):

$$\boldsymbol{V} = [\, \boldsymbol{\sigma_{s,1}} \quad \boldsymbol{\sigma_{s,2}} \quad \dots \quad \boldsymbol{\sigma_{s,m}}\,]. \tag{12}$$

A propósito, Campani e Garcia (2019) testaram um modelo onde o investidor utiliza a média das matrizes de variância-covariância ao invés da média das matrizes de volatilidade, e afirmam que os resultados são virtualmente os mesmos.

As probabilidades de estado na equação (11) dadas por $\boldsymbol{\pi_t}$ são tratadas como variáveis de estado no modelo CGL e seu comportamento é determinado apenas pelos prêmios de risco observados. Assim, assumimos que tais probabilidades adotam o seguinte processo:

$$\begin{bmatrix} d\pi_{1,t} \\ d\pi_{2,t} \\ \cdots \\ d\pi_{m,t} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} \lambda_{i1}\pi_{i,t} \\ \sum_{i=1}^{m} \lambda_{i2}\pi_{i,t} \\ \cdots \\ \sum_{i=1}^{m} \lambda_{im}\pi_{i,t} \end{bmatrix} dt + \begin{bmatrix} \sigma_{11,\pi} & \sigma_{12,\pi} & \dots & \sigma_{1n,\pi} \\ \sigma_{21,\pi} & \sigma_{22,\pi} & \dots & \sigma_{2n,\pi} \\ \cdots & \cdots & \dots & \cdots \\ \sigma_{m1,\pi} & \sigma_{m2,\pi} & \dots & \sigma_{mn,\pi} \end{bmatrix} \boldsymbol{dZ_t}, \tag{13a}$$

com as definições convenientes abaixo:

$$\boldsymbol{d\pi_t} = \begin{bmatrix} d\pi_{1,t} \\ d\pi_{2,t} \\ \cdots \\ d\pi_{m,t} \end{bmatrix}, \quad \boldsymbol{\mu_{\pi,t}} = \begin{bmatrix} \sum_{i=1}^{m} \lambda_{i1}\pi_{i,t} \\ \sum_{i=1}^{m} \lambda_{i2}\pi_{i,t} \\ \cdots \\ \sum_{i=1}^{m} \lambda_{im}\pi_{i,t} \end{bmatrix}, e \;\; \boldsymbol{\sigma_\pi} = \begin{bmatrix} \sigma_{11,\pi} & \sigma_{12,\pi} & \dots & \sigma_{1n,\pi} \\ \sigma_{21,\pi} & \sigma_{22,\pi} & \dots & \sigma_{2n,\pi} \\ \cdots & \cdots & \dots & \cdots \\ \sigma_{m1,\pi} & \sigma_{m2,\pi} & \dots & \sigma_{mn,\pi} \end{bmatrix}, \tag{13b}$$

onde $\boldsymbol{d\pi_t}$ armazena os processos estocásticos seguidos pelas probabilidades $\pi_{i,t}$; $\boldsymbol{\mu_{\pi,t}}$ armazena os *drifts* desses processos para cada regime e $\boldsymbol{\sigma_\pi}$, as volatilidades parciais de cada um desses processos em relação a cada um dos elementos de Wiener que compõem o vetor $\boldsymbol{dZ_t}$. Como definido na seção 3.2, os elementos do vetor $\boldsymbol{dZ_t} = [\, dZ_1 \;\; dZ_2 \dots \; dZ_n]$ são incrementos de processos de Wiener independentes entre si (sem nenhuma perda de generalidade). O processo de estimação da matriz $\boldsymbol{\sigma_\pi}$ encontra-se detalhado no apêndice B.

B. Detalhamento da Estimação dos Parâmetros do Modelo

Para esta aplicação do modelo CGL, assumimos que a periodicidade semanal seria suficientemente curta para razoavelmente igualarmos as probabilidades de transição de Hamilton (1989) com as densidades das probabilidades de transição de nossa aplicação. Ressaltamos que elas seriam idênticas caso tivéssemos considerado períodos infinitesimais (dt).

Com base nos processos apresentados na seção 3.2, temos:

$$Prob\{Y_{t+1} = j \mid Y_t = i\} = P_{ij,\Delta t=1} = P_{ij} = \frac{\lambda_{ij}}{-\lambda_{ii}}\left(1 - e^{\lambda_{ii}}\right),\ i \neq j\,. \tag{14}$$

onde de forma conveniente escolhemos a unidade de tempo com a mesma frequência das informações (uma semana). Assim, obtemos as seguintes identidades:

$$\lambda_{ii} = lnP_{ii} \quad e \quad \lambda_{ij} = \frac{P_{ij} lnP_{ii}}{1 - P_{ii}}. \tag{15}$$

Armazenaremos as probabilidades em tempo discreto (constantes) na matriz $\boldsymbol{P}$:

$$\boldsymbol{P} = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1m} \\ P_{21} & P_{22} & \dots & P_{2m} \\ \dots & \dots & \dots & \dots \\ P_{m1} & P_{m2} & \dots & P_{mm} \end{bmatrix}. \tag{16}$$

Conforme explicado por Hamilton (1989), assumimos que o investidor utiliza as probabilidades filtradas para inferir o regime do período atual e, assim, processar suas decisões ótimas de investimento. Esta inferência considera a iteração das seguintes equações:

$$\hat{\boldsymbol{Y}}_{\boldsymbol{t}|\boldsymbol{t}} = \frac{\hat{\boldsymbol{Y}}_{\boldsymbol{t}|\boldsymbol{t-1}} \odot \boldsymbol{\eta_t}}{\mathbf{1}^T(\hat{\boldsymbol{Y}}_{\boldsymbol{t}|\boldsymbol{t-1}} \odot \boldsymbol{\eta_t})} \quad e \quad \hat{\boldsymbol{Y}}_{\boldsymbol{t+1}|\boldsymbol{t}} = \boldsymbol{P}^T\hat{\boldsymbol{Y}}_{\boldsymbol{t}|\boldsymbol{t}}, \tag{17}$$

onde $\mathbf{1}$ representa aqui um vetor de uns de dimensão $m\,x\,1$ e o símbolo $\odot$ denota uma multiplicação elemento-por-elemento. $\hat{\boldsymbol{Y}}_{\boldsymbol{t}|\boldsymbol{t}}$ e $\hat{\boldsymbol{Y}}_{\boldsymbol{t+1}|\boldsymbol{t}}$, também de dimensões $m\,x\,1$, são vetores que contém as probabilidades filtradas de estar em cada regime, respectivamente nos momentos t e $t+1$, dada a informação atualizada disponível no momento t. Finalmente, $\boldsymbol{\eta_t}$ é outro vetor $m\,x\,1$, cujos elementos são as densidades das probabilidades dos prêmios de risco no momento t condicionadas por estar em cada regime. Para colocar em prática a iteração de $\hat{\boldsymbol{Y}}_{\boldsymbol{t}|\boldsymbol{t}}$ e $\hat{\boldsymbol{Y}}_{\boldsymbol{t+1}|\boldsymbol{t}}$ necessitamos de um ponto de partida. Evidentemente que tal ponto pode ser considerado pelo investidor com base em suas expectativas sobre os estados da economia. Neste estudo, consideramos que o ponto de partida serão as probabilidades de longo prazo, também chamadas de probabilidades ergódigas ou probabilidades incondicionais, definidas por:

$$\hat{\boldsymbol{Y}}_{1|0} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{e_m}, \tag{18}$$

onde $\boldsymbol{e_m}$ denota o último vetor coluna da matriz identidade de ordem $m\,x\,1$ e $\boldsymbol{A}$ é uma matriz m $x\,m$ onde as linhas correspondem às linhas de $\boldsymbol{I_m} - \boldsymbol{P}^T$ ($\boldsymbol{I_m}$ refere-se à matriz identidade de ordem m) e a última linha possui apenas $1's$. A demonstração desta fórmula consta em Hamilton (1989). Assumindo que estamos configurando um modelo de múltiplos regimes de tempo discreto, para encontrar o vetor $\boldsymbol{\eta_{t,i}}$, recordamos o processo seguido pelos ativos de risco, onde admite-se a seguinte solução:

$$\begin{bmatrix} \hat{r}_{1,t} \\ \hat{r}_{2,t} \\ \dots \\ \hat{r}_{n,t} \end{bmatrix} = \begin{bmatrix} a_{1,i} - \frac{1}{2}\sum_{k=1}^{n} \sigma_{1k,i}^2 \\ a_{2,i} - \frac{1}{2}\sum_{k=1}^{n} \sigma_{2k,i}^2 \\ \dots \\ a_{n,i} - \frac{1}{2}\sum_{k=1}^{n} \sigma_{nk,i}^2 \end{bmatrix} dt + \begin{bmatrix} \sigma_{11,i} & 0 & \dots & 0 \\ \sigma_{21,i} & \sigma_{21,i} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \sigma_{n1,i} & \sigma_{n2,i} & \dots & \sigma_{nn,i} \end{bmatrix} \begin{bmatrix} dZ_{1,t} \\ dZ_{2,t} \\ \dots \\ dZ_{n,t} \end{bmatrix}, \qquad (19)$$

tal que o elemento do vetor $\boldsymbol{\eta_{t,i}}$ ocupando a linha i será:

$$\boldsymbol{\eta_{t,i}} = \frac{1}{(2\pi)^{\frac{n+1}{2}} |\boldsymbol{\sigma_i \sigma_i^T}|^{\frac{1}{2}}} exp\left[-\frac{1}{2}(\boldsymbol{LR_t} - \boldsymbol{MLR_i})^T (\boldsymbol{\sigma_i \sigma_i^T})^{-1} (\boldsymbol{LR_t} - \boldsymbol{MLR_i})\right], \qquad (20)$$

onde $\boldsymbol{LR_t}$ é um vetor $n\,x\,1$ de n log-retornos dos prêmios de risco observados no momento t e $\boldsymbol{MLR_i}$ armazena os log-retornos esperados condicionados por regime:

$$\boldsymbol{LR_t} = \begin{bmatrix} \hat{r}_{1,t} \\ \hat{r}_{2,t} \\ \dots \\ \hat{r}_{n,t} \end{bmatrix} \quad \text{e} \quad \boldsymbol{MLR_i} = \begin{bmatrix} a_{1,i} - \frac{1}{2}\sum_{k=1}^{n} \sigma_{1k,i}^2 \\ a_{2,i} - \frac{1}{2}\sum_{k=1}^{n} \sigma_{2k,i}^2 \\ \dots \\ a_{n,i} - \frac{1}{2}\sum_{k=1}^{n} \sigma_{nk,i}^2 \end{bmatrix}. \qquad (21)$$

Os 48 parâmetros acima são estimados através da máxima verossimilhança (MLE) de acordo com o método apresentado por Hamilton (1989). Os últimos parâmetros a serem definidos, de forma independente do procedimento anterior, são os elementos da matriz $\boldsymbol{\sigma_\pi}$, introduzida pelo modelo CGL e definida acima na equação (13b). Para calculá-la, utilizamos $\boldsymbol{D_{s,t}}$ conforme definido na equação (4). Então calculamos os *drifts* semanais ($\boldsymbol{D_{s,t}\pi_t}$) e a matriz de volatilidades ($\boldsymbol{V\pi_t}$), estando os termos $\boldsymbol{\pi_t}$ e $\boldsymbol{V}$ respectivamente definidos nas equações (11) e (12). Assim, a partir da equação (10), estimamos o processo discreto abaixo:

$$\Delta \boldsymbol{Z_t} = \boldsymbol{V\pi_t^{-1}}\left(\boldsymbol{LR_t} - \boldsymbol{D_{s,t}\pi_t}\right), \tag{22}$$

onde $\boldsymbol{LR_t}$ é o mesmo vetor coluna apresentado acima. Utilizamos as equações (13a) e (13b) para escrever a discretização $\left(\Delta\boldsymbol{\pi_t} - \boldsymbol{\mu_{\pi,t}}\right) = \boldsymbol{\sigma_\pi}\Delta\boldsymbol{Z_t}$ e armazenar o lado esquerdo da série temporal em uma matriz $m\ x\ T$ denotada por $(\Delta\boldsymbol{\pi} - \boldsymbol{\mu_\pi})$. Também, armazenamos os termos $\Delta\boldsymbol{Z_t}$ em uma matriz $\Delta\boldsymbol{Z}$ $n\,x\,T$ (onde T corresponde à duração da série temporal). Assim, finalmente, obtemos a estimação desejada:

$$\boldsymbol{\sigma_\pi} = (\Delta\boldsymbol{\pi} - \boldsymbol{\mu_\pi})\Delta\boldsymbol{Z^T}(\Delta\boldsymbol{Z}\Delta\boldsymbol{Z^T})^{-\mathbf{1}} \tag{23}$$

A tabela 4 apresenta os valores da matriz $\boldsymbol{\sigma_\pi}$ para o modelo CGL com 3 regimes e os seguintes ativos de risco: IBOV, PTAX-T-Bill, IMA-B5+ e IMA-B5.

Tabela 4 – Matriz de volatilidade das probabilidades

	Z_1	Z_2	Z_3	Z_4
Regime 1	-3,61%	2,90%	-0,48%	-0,96%
Regime 2	2,95%	-2,33%	1,22%	0,51%
Regime 3	0,65%	-0,56%	-0,74%	0,45%

Nota. A tabela apresenta a matriz $\boldsymbol{\sigma_\pi}$ referente às volatilidades das probabilidades do modelo CGL com 3 regimes e 4 ativos de risco. Os valores correspondem aos dados da última janela de estimação fora da amostra, finda em 04.12.2019.

3. CONSTRAINED PORTFOLIO STRATEGIES IN A REGIME-SWITCHING ECONOMY[7]

3.1. ABSTRACT

We implement an allocation strategy through a regime-switching model using recursive utility preferences in an out-of-sample exercise accounting for transaction costs. We study portfolios turnover and leverage, proposing two procedures to constrain the allocation strategies: a low-turnover control (LoT) and a maximum leverage control (MaxLev). LoT sets a dynamic threshold to trim minor rebalancing, reducing portfolio turnover, mitigating costs. MaxLev calculates dynamic adjustments to the risk aversion parameter to constrain the portfolio leverage. The MaxLev adjustments depend on the risk aversion and permitted portfolio leverage, which enables optimal strategies considering the leverage constraints. The study uses US equity portfolios, and shows that, first, models with LoT result in superior return-to-risk measures than those without it when transaction costs increase. Second, considering transaction costs, the return-to-risk measures of the models using MaxLev closely match or exceed those from the corresponding unconstrained regime-switching benchmarks. Third, MaxLev returns have lower volatility and higher return-to-risk than conventional numerically constrained benchmarks. Fourth, the certainty equivalent returns indicate that models using MaxLev and LoT outperform both single-state models and unconstrained regime-switching models with statistical significance.

[7] This is section presents the second article of the Thesis. It was accepted in 2022 and the publication is forthcoming at the Financial Markets and Portfolio Management (FMPM): https://link.springer.com/article/10.1007/s11408-022-00414-x.

3.2 INTRODUCTION

According to Munk (2013), recursive utility functions, as discussed by Epstein and Zin (1989), became popular among practitioners due to a superior accuracy in characterizing the investor's preferences. Given that lifetime utility at time t can be captured by a utility index referred to in the literature as felicity, previous dynamic functions like the power utility and the exponential utility considered that investor's felicity was time-additive, meaning that it would be merely incremental. Unlike them, recursive functions consider that tomorrow's felicity also depends on today's felicity, enabling investors to time the uncertainty resolution, which is reflected through the disentanglement of the attitudes toward atemporal risks (relative risk aversion) from the attitudes toward shifts in consumption over time (elasticity of intertemporal substitution). One function that captures such recursive preferences in continuous time is the stochastic differential utility introduced by Duffie and Epstein (1992).

To the best of our knowledge, the asset allocation model presented by Campani, Garcia, and Lewin (2021), herein denoted the CGL model, is the only model to apply the stochastic differential recursive utility function in a regime-switching framework. It consists of an approximate analytical method, which the authors demonstrate to be sufficiently accurate to solve the allocation problem. Before this model, the literature solved dynamic allocation under multiple regimes with the power utility function, via numeric methods such as the Monte Carlo simulation – Sass and Haussmann (2004), Guidolin and Timmermann (2007), and Liu (2011).

Lewin and Campani (2020a,b) test the CGL model in different settings, sharing the same finding: the CGL returns consistently outperform reference portfolios. So far, the CGL returns have been presented in the scope of unconstrained strategies and without accounting for transaction costs, which is not unusual in the literature. For example, Ang and Bekaert (2002) and Graflund and Nilsson (2003) present optimal solutions without accounting for costs.

Our study assesses the impacts of costs and the strategy's weights in the CGL model implementation. We endorse the literature that strategies might become unpractical if aspects such as portfolio leverage and rebalancing were not constrained. Thus, our objective is to propose filters to constrain the CGL portfolio's maximum leverage and to lower its turnover.

We propose a maximum leverage control, referred to as the MaxLev filter. Analogously to the drawdown control by Nystrup, Boyd, Lindström, and Madsen (2019), MaxLev dynamically adjusts the risk aversion parameter (γ) to constrain the maximum leverage. First, it collects the optimal portfolio leverage for a theoretical 100% probability for each regime. Then, it calculates the adjustments over γ to confine the optimal leverage inside the maximum leverage permitted for each regime, weighting them by the out-of-sample probabilities. Although the adjustments originate from the leverage limits, they remain dependent on γ. Thus, MaxLev presents a constraining method constantly proportional to γ. It distinguishes from other numerical methods that just trim excessive weights, in which the constrained leverage might become flattened at the level of the limits if the base case is often excessively leveraged. With MaxLev, leverage is constantly proportional to the regimes' probabilities, as expected from an unconstrained allocation. Therefore, the risk exposure is balanced by the regimes' expectations, enabling an optimal strategy that remains inside the leverage limits at each point in time.

We also propose a low-turnover control (LoT) to rebalance the portfolio if the turnover from t to $t+1$ is above a threshold. The rebalancing policy is dynamically recalculated, updating the threshold by the historic turnover during significant probability changes.

The study compares the CGL model with and without MaxLev and LoT in different levels of transaction costs using single-state and regime-switching benchmarks in an out-of-sample exercise. The empirical results show that MaxLev matches or improves the return-to-risk measures relative to the benchmarks, while LoT produces economic value when cost levels are higher. The certainty equivalent returns indicate that the CGL model using MaxLev and LoT outperform most benchmarks with statistical significance.

3.3 LITERATURE REVIEW

Ang and Timmermann (2012) have reviewed the literature since the seminal paper by Hamilton (1989). The authors indicate the increasing importance of regime-switching models in asset allocation strategies, where the power utility functions have been widely used. For example, Ang and Bekaert (2002) and Guidolin and Timmermann (2007, 2008) numerically solved the problem using the constant relative risk aversion utility (CRRA). Later, Guidolin and Hyde (2012) applied the same procedure while Çanakoğlu and Özekici (2012) found the explicit solution for maximizing the expected utility of terminal wealth with a hyperbolic absolute risk aversion function (HARA), but suggest that further studies should involve other utility functions. In turn, Ang and Bekaert (2002) state that the recursive utility functions improve regime-switching effects in allocation problems.

Using the recursive utility, Kraft, Seiferling, and Seifried (2017) and Xing (2017) solved the consumption-investment optimization with Epstein and Zin's (1989) function, but neither performed it observing regimes. For such a case, Campani, Garcia, and Lewin (2021) presented a regime-switching allocation strategy (CGL model) in continuous time based on Duffie and Epstein's (1992) stochastic differential recursive utility function. They offered an approximate analytical solution based on Campani and Garcia (2019), given a system of partial differential equations. Campani, Garcia, and Lewin (2021) and Lewin and Campani (2020a,b) present optimal allocations based on the CGL model. We advance the research based on such a model accounting for transaction costs and leverage and turnover constraints – a gap left open.

The literature presents several approaches to constrain portfolio leverage, indicating there is no consensus. One is to shrink the covariance matrix toward an identity matrix. Ledoit and Wolf (2004) and Fiecas, Franke, Von Sachs, and Tadjuidje (2017) apply it to constrain mean-variance portfolios. In another approach considering the quadratic utility, Clarke, De Silva, and Thorley (2002) and DeMiguel, Garlappi, Nogales, and Uppal (2009) use a factor to shrink the weight vector. Later, Dal Pra, Guidolin, Pedio, and Vasile (2018) simply prevent the weights exceeding leverage limits using power utility preferences. These approaches, however, have significant limitations. In the first approaches, the problem of modifying high dimensional covariance matrices becomes more pronounced for regime-switching models (Nystrup, Boyd, Lindström, and Madsen, 2019). In the latter, if leverage is often excessive, the constrain flattens it at the boundaries level, unbalancing the risk exposure to the regimes' expectations.

Inspired by Nystrup, Boyd, Lindström, and Madsen (2019) drawdown control, we overcome such limitations constraining leverage around the premise of a dynamically adjusted γ, the risk aversion parameter. The authors increase γ in the domain of losses, as a reaction mechanism. However, in case of successive losses, it traps the investor in elevated risk settings.

Although we also calculate dynamic adjustments for γ, the investor's risk preferences do not disconnect from the base case (γ) in our method. Thus, γ remains an unobservable behavioral parameter. Based on the maximum leverage permitted, we only infer adjustments to γ, assuming the leverage limits might be conditioned on the regimes' expectations. We emphasize that we do not set γ according to the regimes; hence we do not shape the utility function – in a behavioral sense – as regime dependent. Instead, we consider that the leverage limits impact the utility function. Therefore, the method allows an optimal portfolio strategy that respects the leverage constraints at each point in time.

In addition, the literature mitigates transaction costs by imposing rebalancing policies that constrain portfolio turnover. For example, Lunde and Timmermann (2004) and Guidolin and Timmermann (2008) timed the regimes' duration to separate long and short-run movements. Instead, we follow Clarke, De Silva, and Thorley (2002), defining a rebalancing policy upon a minimum turnover threshold (limit). However, while they consider a static threshold, we dynamically compute it based on regimes' probabilities. Thus, we maintain an updated rebalancing policy.

3.4 METHOD

3.4.1 THE REGIME-SWITCHING ECONOMY

We apply the strategy developed by Campani, Garcia, and Lewin (2021) with a continuous-time regime-switching model in which investors maximize their stochastic differential recursive utility functions through optimal portfolio strategies.

State variable. We assume a regime-switching economy governed by the unobservable state variable Y_t representing an independent right-continuous-time Markov chain process admitting only values in $R = \{1, 2, \dots, m\}$, where R is a finite set of m

possible regimes. Following Hamilton (1989), the state variable behavior is modeled using transition probabilities that rule if the economy will remain at the same regime or jump to a new one after an exponentially distributed length of time. If i is the current regime and λ_{ij} is the density of transition probabilities between regimes i and j, the probability to jump to another regime j at time Δt is:

$$P_{ij,\Delta t} = \frac{\lambda_{ij}}{\sum_{k \neq i} \lambda_{ik}} \left(1 - e^{-\sum_{k \neq i} \lambda_{ik} \Delta t}\right), \text{ with } j \neq i \in R \text{ and } \lambda_{ij} > 0, \tag{1}$$

where $\lambda_{ii} = -\sum_{k \neq i} \lambda_{ik} \leq 0$ such that $P_{ij,\Delta t} = \frac{\lambda_{ij}}{-\lambda_{ii}} \left(1 - e^{\lambda_{ii} \Delta t}\right)$. Hence, the probability of staying at the same regime i over the next Δt is given by $P_{ii,\Delta t} = e^{\lambda_{ii} \Delta t}$, with λ_{ij} assumed constant.

Assets dynamics. As Lewin and Campani (2020a,b), we estimated the model for excess returns ($\hat{r}$) over the riskless asset (r_f). We consider that the risk-premia from the n risky assets are defined through the following multidimensional stochastic process:

$$\begin{bmatrix} d\hat{r}_{1,t} \\ d\hat{r}_{2,t} \\ \dots \\ d\hat{r}_{n,t} \end{bmatrix} = \boldsymbol{\mu}_{s,t}\, dt + \boldsymbol{\sigma}_{s,t}\, \boldsymbol{dZ}_t = \begin{bmatrix} \mu_{1,t} \\ \mu_{2,t} \\ \dots \\ \mu_{n,t} \end{bmatrix} dt + \begin{bmatrix} \sigma_{11,t} & 0 & \dots & 0 \\ \sigma_{21,t} & \sigma_{22,t} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \sigma_{n1,t} & \sigma_{n2,t} & \dots & \sigma_{nn,t} \end{bmatrix} \begin{bmatrix} dZ_{1,t} \\ dZ_{2,t} \\ \dots \\ dZ_{n,t} \end{bmatrix}, \tag{2}$$

where $\boldsymbol{\mu}_{s,t}$ is a $n \times 1$ vector of the instantaneous expected risk-premia (drifts), $\boldsymbol{\sigma}_{s,t}$ is an $n \times n$ lower triangular volatility matrix, and $\boldsymbol{dZ}_t$ is a column vector with n increments of independent standard Wiener processes. Note that both $\boldsymbol{\mu}_{s,t}$ and $\boldsymbol{\sigma}_{s,t}$ are time-varying and conditioned by the state variable Y_t. In turn, with $Y_t = i,\ i \in R$, we find:

$$\boldsymbol{\mu}_{s,t} = \boldsymbol{\mu}_{s,i} = \begin{bmatrix} \boldsymbol{\mu}_{1,i} \\ \boldsymbol{\mu}_{2,i} \\ \dots \\ \boldsymbol{\mu}_{n,i} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\sigma}_{s,t} = \boldsymbol{\sigma}_{s,i} = \begin{bmatrix} \sigma_{11,i} & 0 & \dots & 0 \\ \sigma_{21,i} & \sigma_{22,i} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \sigma_{n1,i} & \sigma_{n2,i} & \dots & \sigma_{nn,i} \end{bmatrix}, \tag{3}$$

in which, $\boldsymbol{\mu}_{j,i}$ coefficients and $\boldsymbol{\sigma}_{s,i}$ matrices are constant for each $j = \{1, 2, \dots, n\}$. We underline that $\boldsymbol{\sigma}_{s,i}$ elements are defined as partial volatilities, e.g., $\sigma_{21,i}$ denotes partial volatility of asset 2 in relation to the first Wiener process ($dZ_{1,t}$) in regime i, and also that $\boldsymbol{\sigma}_{s,i}\boldsymbol{\sigma}_{s,i}^T$ represent the regime-dependent variance-covariance matrix. Still, as the drifts are regime-dependent and simultaneously time-dependent, it means that they can vary in time even if the regime remains unchanged. Such drifts are stored in an $n \times m$ drift matrix:

$$\boldsymbol{D}_{s,t} = \begin{bmatrix} \boldsymbol{\mu}_{1,1} & \boldsymbol{\mu}_{1,2} & \dots & \boldsymbol{\mu}_{1,m} \\ \boldsymbol{\mu}_{2,1} & \boldsymbol{\mu}_{2,2} & \dots & \boldsymbol{\mu}_{2,m} \\ \dots & \dots & \dots & \dots \\ \boldsymbol{\mu}_{n,1} & \boldsymbol{\mu}_{n,2} & \dots & \boldsymbol{\mu}_{n,m} \end{bmatrix}. \tag{4}$$

Provided such assets and state variable processes, we obtain the regime-parameters through the maximum likelihood (ML) estimation methodology. These parameters are: the drift matrix $\boldsymbol{D}_{s,t}$, the volatility matrix $\boldsymbol{\sigma}_{s,i}$ and the transition probabilities $P_{ij,\Delta t}$ (where $j \neq i \in R$). In fact, the number of parameters to be estimated is $[mn + mn(n+1) \div 2 + m(m-1)]$. Given the unobservable nature of the regimes, we assume that investors can infer the regimes' occurrences through filtered probabilities observing the assets' past returns. We detail the probabilities estimation according to Hamilton's (1989) procedure in the supplementary file.[8]

[8] The supplementary file is found in: https://link.springer.com/article/10.1007/s11408-022-00414-x.

3.4.2 THE PORTFOLIO STRATEGY

After configuring the regime-switching economy, we address the portfolio strategy. First, considering W_t as the wealth in t and $\boldsymbol{\alpha_t}$ as the $1 \times n$ vector of portfolio shares of the risky assets, and $(1 - \boldsymbol{\alpha_t 1})$ as the riskless asset share, wealth dynamics can be expressed as:

$$dW_t = (1 - \boldsymbol{\alpha_t 1}) W_t \, r_f \, dt + W_t \, \boldsymbol{\alpha_t} \frac{dS_t}{S_t}$$
$$= W_t \, r_f \, dt + W_t \, \boldsymbol{\alpha_t} [\boldsymbol{D_{s,t}} \, \boldsymbol{\pi_t} \, dt + (\boldsymbol{V \pi_t}) \, \boldsymbol{dZ_t}], \tag{5}$$

where $\mathbf{1}$ is a column vector of n ones, $\frac{dS_t}{S_t}$ is the column vector with n infinitesimal risky asset returns, $\boldsymbol{\pi_t}$ is a column vector with the m filtered probabilities in t and $\boldsymbol{V}$ is an $1 \times m$ row vector containing the regime-dependent covariance matrices ($\boldsymbol{\sigma_{s,i}}$).

Utility function. In the CGL model, the investor's preferences are characterized as continuous-time and modeled by the stochastic utility function from Duffie and Epstein (1992):

$$J_t = E_t \left[\int_{u=t}^{T} f(C_u, J_u) \, du + \frac{W_T^{1-\gamma}}{1-\gamma} \right], \tag{6}$$

where E_t is the expected value in the current moment (t); T is the investment horizon; f is the recursive aggregator of the utility function J_t in function of consumption rate C_u (in moment u) and J_u, the continued utility in u. In turn, W_T is the investor's terminal wealth while γ is the risk aversion coefficient. The following function details the utility function aggregator:

$$f(C,J) = \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)J\left\{\left[\frac{C}{[(1-\gamma)J]^{\frac{1}{1-\gamma}}}\right]^{1-\frac{1}{\psi}} - 1\right\}, \tag{7}$$

where β is the time preference rate of the investor's utility (felicity) and ψ is the elasticity of intertemporal substitution, i.e., consumption choices over time. Thus, we must set ψ, β and γ to configure the strategy using recursive utility.

Campani and Garcia (2019) analyze the sensitivity of consumption and portfolio choices over the value of both preference parameters γ and ψ for a problem similar to Campani, Garcia, and Lewin (2021) but in a single-state model. They indicate that the value of ψ affects consumption preferences but barely affects the allocation strategy. Later, Campani, Garcia, and Lewin (2021), considering regime-switching models, conclude that the impact of consumption-to-wealth ratio variations is minimal in the allocation strategy. Thus, we simplify the current application disregarding intermediary consumption. Considering that $\psi > 1$ is this case where substitution effects dominate and the investor is willing to postpone consumption, we define $\psi = \infty$ to represent the investor that waits for the terminal horizon to consume their wealth. Then, given a problem without intermediary consumption, the value of β does not significantly affect the allocation strategy. Campani, Garcia, and Lewin (2021) and Guidolin and Timmermann (2007) also show that the investment horizon has a negligible impact on the allocation strategy considering frequent rebalancing, like in our application. Following them, we consider $\gamma = 5$ our base case.

Campani, Garcia, and Lewin (2021) demonstrate that the general solution quantifying the investor total optimal utility in t ($V_t = \sup J_t$) admits the wealth-separable solution:

$$V(W_t, \boldsymbol{\pi_t}, \tau) = H(\boldsymbol{\pi_t}, \tau)\ \frac{W_t^{1-\gamma}}{1-\gamma}\ , \tag{8}$$

where $\tau = T - t$ is the time until the final horizon and $H(\boldsymbol{\pi_t}, \tau)$ is a function in terms of the time to horizon and the regimes' probabilities vector. But, as an exact analytical expression for $H(\boldsymbol{\pi_t}, \tau)$ is not yet available in the literature, Campani, Garcia, and Lewin (2021), based on the the Bellman equation solve the problem by offering the following approximate analytical expression:

$$H(\boldsymbol{\pi_t}, \tau) = exp\left[A_0(\tau) + \sum_{i=1}^{m} A_i(\tau)\, \pi_{i,t} + \sum_{i=1}^{m} B_i(\tau)\pi_{i,t}^2 + \sum_{j<i} C_{ij}(\tau)\, \pi_{i,t}\, \pi_{j,t}\right], \tag{9}$$

where $\pi_{i,t}$ is regime i probability at time t. Meanwhile, $A_0, A_i, B_i,$ and C_{ij} are time-horizon coefficients obtained from the solution of the Bellman equation under a system of partial differential equations (PDE). The supplementary file provides details for solving the Bellman equation, while Campani, Garcia, and Lewin (2021) demonstrate the PDE.

Portfolio weights. Given the approximate solution for $V(W_t, \boldsymbol{\pi_t}, \tau)$ and the coefficients $A_0, A_i, B_i,$ and C_{ij} from function $H(\boldsymbol{\pi_t}, \tau)$, the CGL model presents the optimal weights for the regime-switching allocation using the recursive utility function given by the following form:

$$\begin{aligned} \boldsymbol{\alpha_t} =\ & \frac{1}{\gamma}\ \left(\boldsymbol{D_{s,t}}\, \boldsymbol{\pi_t}\right)^T [(\boldsymbol{V\pi_t})(\boldsymbol{V\pi_t})^T]^{-1} \\ & + \tfrac{1}{\gamma}\textstyle\sum_{i=1}^{m}\left[A_i(\tau) +\ 2B_i(\tau)\, \pi_{i,t} +\ \sum_{j\neq i} C_{ij}(\tau)\, \pi_{j,t}\right] \boldsymbol{\sigma_{i,\pi}}\, (\boldsymbol{V\pi_t})^{-1}\ , \end{aligned} \tag{10}$$

where $\boldsymbol{\alpha_t} = [\boldsymbol{\alpha_{1,t}} \quad \cdots \quad \boldsymbol{\alpha_{n,t}}]$, $\boldsymbol{\sigma_{i,\pi}} = [\sigma_{i1,\pi} \quad \sigma_{i2,\pi} \quad \dots \quad \sigma_{in,\pi}]$, $i \in R$ and $j = \{1, 2, \dots, n\}$.

Maximum leverage control (MaxLev). After determining the unconstrained optimal weights, we must ensure that the strategy is confined within the maximum leverage permitted. To constrain it by the maximum leverage levels, we propose dynamically adjusting the risk parameter γ . We emphasize that γ is not set according to the regimes (as the parameter is unobservable). We limit leverage by calculating adjustments to the base case γ, obtaining an optimal portfolio strategy that preserves such a maximum leverage policy.

It contrasts to conventional numerically constrained (NC) methods that prevent the weights from exceeding leverage limits by trimming excessive weights. For example, NC procedures might flatten the portfolio leverage at the policy limits if the base case frequently leads to excessive leverage. Instead, using MaxLev, leverage is constantly proportional to the regimes' probabilities and the risk exposure, balanced to the regimes' expectations.

We firstly collect the unconstrained weights at 100% probability for each regime through equation (10) with $\boldsymbol{\hat{\pi}_i} = i^{th}$ column of an m-order identity matrix. We then compute leverage as the sum of positive weights that exceeds 100% of the portfolio shares, storing in a $1 \times m$ vector the unconstrained leverages conditioned by each regime ($\boldsymbol{UncLev}$), whose elements are:

$$UncLev_i = \left[\sum_{k=1}^{n+1} \max\left(\boldsymbol{\hat{\alpha}_{\hat{\pi}_i,k}}, 0\right)\right] - 1, \text{ with } i \in R = \{1, 2, \dots, m\}, \tag{11}$$

where $\hat{\boldsymbol{\alpha}}_{\hat{\boldsymbol{\pi}}_i,\boldsymbol{k}}$ is the k^{th} element of the row vector $\hat{\boldsymbol{\alpha}}_{\hat{\boldsymbol{\pi}}_i} = \left[\boldsymbol{\alpha}_{\hat{\boldsymbol{\pi}}_i} \quad \left(1 - \boldsymbol{\alpha}_{\hat{\boldsymbol{\pi}}_i}\mathbf{1}\right)\right]$ formed by the risky asset weight vector along with the riskless asset weight, all conditioned by $\hat{\boldsymbol{\pi}}_{\boldsymbol{i}}$. So, to build the leverage policies, we assume that investors' risk appetite adapts according to the economic states, e.g., in a shift from crash to rally states, pessimistic investors may become more optimistic and prone to set leverage at a different level. Therefore, we define maximum leverage values ($L_{b,i}$) conditioned by regime $i \in R$, where we also consider the possibility to build z leverage policies, with z as a positive integer representing the number of policies investigated at the research and $b = \{1, 2, \dots, z\}$. The policies are then collected in a $z \times m$ matrix as follows:

$$\boldsymbol{MaxLev} = \begin{bmatrix} L_{11} & \cdots & L_{1m} \\ \vdots & \ddots & \vdots \\ L_{z1} & \cdots & L_{zm} \end{bmatrix}. \tag{12a}$$

The method enables investors to assign regime-conditioned leverage policies to meet their market expectations, although we do not explore this idea. The z leverage policies (limits) adopted in the research are as follows (this range of values is justified in section 3.4.4):

$$\boldsymbol{MaxLev} = \begin{bmatrix} 300\% & 300\% & 300\% & 300\% \\ 200\% & 200\% & 200\% & 200\% \\ 100\% & 100\% & 100\% & 100\% \\ 0\% & 0\% & 0\% & 0\% \end{bmatrix}. \tag{12b}$$

We advance by calculating adjustments for γ to confine $UncLev_i$ within limits stated in $\boldsymbol{MaxLev}$, with which we infer the base case risk parameter ($\gamma = 5$) conditionally to the limits and regimes. The procedure emerges on a new $z \times m$ matrix whose elements are:

$$\gamma_{b,i} = \gamma \times max\,[\,(1 + UncLev_i) \div (1 + L_{b,i})\,,\ 1\,]\,, \tag{13}$$

where the maximum operator preserves the original value of γ when the element $UncLev_i$ is already below the limit imposed by $L_{b,i}$. Multiplying the new matrix b^{th} row (expressed by $\boldsymbol{\gamma_b} = [\gamma_{b,1} \quad \cdots \quad \gamma_{b,m}]$) by the column vector of regimes probabilities at time t ($\boldsymbol{\pi_t}$), we obtain a dynamic value as our risk parameter, which is an average of $\boldsymbol{\gamma_b}$ (dynamically) weighted by $\boldsymbol{\pi_t}$. Plugging it on equation (10), we then find the weights constrained by policy b at time t:

$$\begin{aligned}\boldsymbol{\alpha_{b,t}} = \frac{1}{(\boldsymbol{\gamma_b \pi_t})} \left(\boldsymbol{D_{s,t}\, \pi_t}\right)^T [(\boldsymbol{V\pi_t})(\boldsymbol{V\pi_t})^T]^{-1} \\ + \frac{1}{\gamma} \textstyle\sum_{i=1}^{m} \left[A_i(\tau) + 2B_i(\tau)\, \pi_{i,t} + \sum_{j\neq i} C_{ij}(\tau)\, \pi_{j,t}\right] \boldsymbol{\sigma_{i,\pi}}\, (\boldsymbol{V\pi_t})^{-1}.\end{aligned} \tag{14}$$

3.4.3 THE APPLICATION OF THE CGL MODEL

Regime-switching models may offer more hedged positions than single-state models, given the regimes' expectations. Still, such protection might lead to peaks of excessive leverage in particular conditions, such as high correlations in less uncertain states. Thus, we study a single class portfolio to increase the chances of observing this effect for controlling it.

Data set. We allocate $n = 3$ risky assets along with the risk-free asset. The risky assets are the first, third, and fifth quintiles from the value-weighted daily returns of the portfolios formed on size by Kenneth French. We will denote them as small, mid, and large caps, respectively. Along with such a US equity portfolio, the risk-free asset is the

3-month Treasury bill secondary market rate extracted from the Board of Governors of the Federal Reserve System. We converted the data series to weekly observations as Lewin and Campani (2020a,b), given that a daily frequency could defy the application feasibility, and a monthly frequency would limit the amplitude of the out-of-sample exercise. Thus, the data set encompasses 3552 weekly observations, starting on January 8th, 1954, and ending on November 30th, 2021.

The out-of-sample exercise. We organized the weekly observations in 174 rolling windows (three widows per year). Following DeMiguel, Garlappi, and Uppal (2009) and Bulla, Mergner, Bulla, Sesboüé, and Chesneau (2011), we set them with approximately ten years of data. Then, the regime parameters were (re)estimated via ML with observations prior to the first window date and held during the four subsequent months. Meanwhile, at every new week, we define the strategy from the filtered probabilities estimated in t for $t + 1$. Hence, replicating only the available information at the investment decision moment, this procedure delivers out-of-sample returns. Thus, the exercise extends from January 3rd, 1964, to November 30th, 2021.

Number of regimes. To define the number of regimes (m), we consider $m = 1$ as the single regime model presented in the results section. Meanwhile, $m = 4$ is the highest number of states often observed in the literature. For example, Guidolin and Timmermann (2008) identify four regimes in a study with US equity indices. In addition, Guidolin and Ono (2006) indicate a saturation ratio between the number of estimated parameters and the data series length. They work with ratio values of around 30. Given the rolling window length, in our application, only models with $m \leq 4$ present ratios above 30. Hence, we tested models with $m = 2, 3, 4$. Table 1 shows their information criteria (IC), in which we present 3 out of the 174 (re)estimations obtained using rolling windows: 1, 87, and 174. The differences between the models' IC are relatively stable during the (re)estimations. Therefore, as Table 1 indicates, $m = 4$ dominates the other models; thus, we set the application with four regimes.

Table 1 – Information criteria

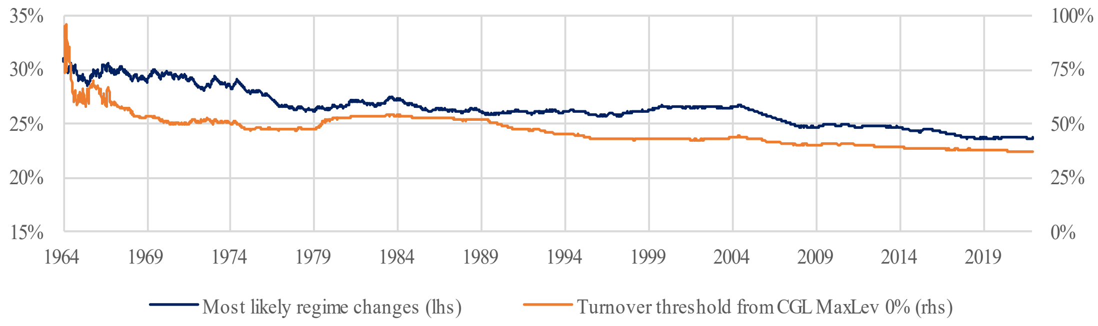
m	Oldest rolling window			Intermediate rolling window			Most recent rolling window		
	AIC	**BIC**	**H-Q**	**AIC**	**BIC**	**H-Q**	**AIC**	**BIC**	**H-Q**
2	-10.373	-10.359	-10.396	-9.631	-9.617	-9.654	-9.000	-8.986	-9.023
3	-10.416	-10.393	-10.453	-9.694	-9.671	-9.732	-9.027	-9.004	-9.065
4	-10.457	-10.424	-10.512	-9.757	-9.723	-9.811	-9.043	-9.010	-9.098

Notes. The table indicates the information criteria for the models with $n = 3$ risky assets under 2, 3, and 4 regimes. The columns present Akaike (AIC), Bayes-Schwartz (BIC), and Hannan-Quinn (H-Q) for three rolling windows from the out-of-sample exercise: the oldest (May/1954-Dec/1963), the intermediary (May/1983-Dec/1992), and the most recent (Jan/2012-Aug/2021). The full out-of-sample exercise was performed with 174 rolling windows.

Low-turnover control (LoT). We define turnover as the percentage of the total portfolio value to be exchanged for new positions. Guidolin and Timmermann (2008) affirm that weights of single regime strategies differ from multi-regimes as the latter capture the short-run market timing effects, while slower long-run movements drive the former. Hence, rapid state shifts are expected to create higher turnover than single regimes strategies, underlining the urgency of restricting turnover to manage costs. Our rebalancing policy consists of two parts. First, we filter out total portfolio turnovers below a given threshold. We then eliminate portfolio changes that occur due to minor updates of the probabilities. Second, we assure that the weights remain close to the most updated strategy imposing a minimum of one monthly rebalancing.

The LoT filter informs whether the best decision is to rebalance at every moment in which the investor needs to decide on the optimal allocation strategy (i.e., weekly). LoT recommends rebalance if the optimal strategy generated by the model at the decision-making moment is significantly different from the current position (otherwise, it recommends holding the current position). We are not computing the turnover between optimal weights from the previous and the current weeks, but the previous week's optimal weights impacted by the market movements (from the current week) and the current week optimal weights.

Figure 1 – Most likely regime changes dynamically adapting the threshold



Notes. We show the most likely regime changes and their effects over the turnover threshold, illustrated by CGL MaxLev 0%. The most likely regime is given by the highest filtered probability at each moment (hence, it is independent from the allocation strategy). We accrue the changes when the most likely regime alters from $t-1$ to t. The percentage of most likely regime changes (c) is the historic number of changes relative to the number of observations. We started these observations in 1959 to present the data of the out-of-sample exercise, from January 3rd, 1964 to November 30th, 2021, discarding the first five years to avoid initial noise.

The rebalancing decision is made given a threshold that defines a minimum turnover limit. Clarke, De Silva, and Thorley (2002) applied a process with a threshold, but they consider it static. In its turn, LoT dynamically adapts it – thus constantly updating the rebalancing policy. The process is as follows. At each rebalancing decision, we observe the percentage of most likely regime changes (c) as given by the filtered probabilities, and the optimal total portfolio turnovers considering past historical data. We assume that c is a proxy for the number of probabilities updates that shall be considered. We now define $u = (1-c)$ and calculate the u^{th} percentile at the historical dataset of optimal portfolio turnovers: this will be the minimum accepted turnover to define whether there will be rebalancing. As such, the threshold will recommend rebalancing only upon significant turnovers and, on average, at the same rate as the historically most likely regime changes. Figure 1 illustrates the effects of LoT filtering.

Figure 1 shows the most likely regime changes (c) representing the dynamic rebalancing policy and indicates that significant probabilities updates occur 24% to 31% of the time. In 1979-2004 and 2009-2021, there are plateaus of c at approximately 26% and 24%, respectively. In 2004-2009, c reduces given a long period under only one regime, as the results section will show. We exemplify these effects over CGL MaxLev 0%. In 1979-2004 and 2009-2021, its turnover threshold spans from 42% to 54% and 37% to 41%, respectively. Meanwhile, it extends from 27% to 96% in the full sample. Thus, it dynamically adapts as time passes.

Transaction Costs. We assume the investor accounts for transaction costs. We studied Bulla, Mergner, Bulla, Sesboüé, and Chesneau (2011), Gârleanu and Pedersen (2013), and Nystrup, Boyd, Lindström, and Madsen (2019), who fixed transaction costs at 10 basis points (0.10%) for dynamic asset allocation for daily trading. The last authors additionally propose to account for holding costs, charged at the risk-free rate over the short sales. In our application, holding costs are underlying the model, as we assume that the investor borrows at the risk-free rate for short selling. Nevertheless, to be conservative and account for potential inefficiencies, like illiquidity costs, we present the returns for higher costs. Therefore, we will present the exercise considering the following levels of transaction costs: 0.10%, 0.20%, and 0.40%.

3.4.4 BENCHMARKS

The out-of-sample exercise presents the net returns from the proposed model (CGL MaxLev) to constrain leverage at zero, 100%, 200%, and 300% levels. At these leverage levels, the CGL MaxLev volatilities are (approximately) restricted by the volatility of the investigated assets, as indicated in section 3.5.3. CGL MaxLev is compared to four models as follows. First, two models without regimes: the equal-weights and the single regime model. Then, two regime-switching models: the unconstrained CGL model and the numerically constrained CGL model. The regime-switching models are presented with and without the LoT filter. Every case considers the four levels of transaction costs to compare different cost perspectives.

Equal-weights portfolio ($\mathbf{1/n}$). DeMiguel Garlapi, and Uppal (2009) demonstrate that a $1/n$ portfolio, despite being a naïve strategy, outperforms a number of dynamic models based on optimal rules. Thus, it represents a common benchmark.

Single regime model (SR). The SR model corresponds to the recursive utility preferences of an investor who does not account for a multiple regime economy. We use it to assess the impact of the regime-switching on the overall performance.

Unconstrained CGL model (UNC). We compute the unconstrained CGL model as in the original CGL model from Campani, Garcia, and Lewin (2021), without any adjustments on γ (i.e., fixed preferences). Later, we empirically observe that the maximum leverage under such a setup is not distant from 100%, 200%, and 300% when γ is 75, 100, and 200, respectively. This comparison reveals the impacts from the proposed filtering process (MaxLev) versus the original application, in which we solve the problem without constraints.

Numerically constrained CGL model (NC). An alternative constraining approach is simply preventing the weights from exceeding leverage limits recalculating them proportionally to the constraint when the optimal leverage exceeds the maximum permitted. We apply the NC procedure at the leverage levels: zero, 100%, 200%, and

300%. This model benchmarks CGL MaxLev relative to a constrained portfolio using fixed preferences.

Other benchmarks. We calculated the exercise for the power utility, as Guidolin and Hyde (2012). It is a particular case from our codes, obtained when $\psi = 1/\gamma$. However, the results from the power utility are very close to those of the recursive utility, given $\psi = \infty$, as expected. It occurs because the value of ψ impacts consumption, but it minimally impacts the allocation strategy (Campani and Garcia, 2019). In addition, Campani, Garcia, and Lewin (2021) show that the impact of consumption-to-wealth ratio variations is also minimal in the allocation model. Thus, both results converge as the investor postpones consumption to the final horizon by setting $\psi = \infty$. The power utility results are indicated in the supplementary file.

3.4.5 RESULTS TESTS

Following Fugazza, Guidolin, and Nicodano (2015) and Campani, Garcia, and Lewin (2021), we use the annualized certainty equivalent returns (CER) to compare and rank different strategies. The authors provide the derivation of the following expression used to compute CER:

$$CER_i(\gamma, t) \equiv \frac{F}{T}\left\{\frac{1}{W_t}\left[\frac{1}{K-T}\sum_{\tau=1}^{K-T}\left[W_{\tau+T}\left(\widehat{\omega}_{i,t}(\gamma, T)\right)\right]^{\frac{1}{1-\gamma}}\right]-1\right\}, \tag{15}$$

where F is the data frequency (52 weeks per year), T is the horizon (520 weeks), K is the number of out-of-sample returns, $\widehat{\omega}_{i,t}$ are the proportions of the wealth invested in asset i, and W_t is the initial wealth (set to 1).

The next section presents the CER differences to compare two portfolios. Below such figures, it will also report the 95% bootstrap confidence intervals drawn from 1,000,000 samples with replacement, using the bias-corrected and accelerated percentile method due to non-normalities in the out-of-sample returns.

3.5 RESULTS

3.5.1 THE FOUR-REGIME MODEL

We analyze the estimated parameters in Table 2. Panel A indicates the single-state parameters as a reference of estimates that do not account for regime-switching. Panels B.1 and B.2 show the regimes' parameters with the characteristics of four economic states: crash, bear market, bull market, and rally. The regimes are interpreted as follows.

In the crash state, expected returns are highly negative and volatile, thus uncertainty is exceedingly high. However, after a crash regime, the transition probabilities reveal similar chances of shifting to any of the three less uncertain states, indicating that recovery does not necessarily follow it. On average, the crash and the bear market's duration are 10 weeks, while the bull market lasts 28 weeks. Therefore, volatilities in the bull market converge with those from the single-state, although the bull market's returns are slightly higher than the single-state. In turn, the rally state presents the highest returns and the lowest volatilities and correlations.

The crash state, bear, and bull markets show high correlations, from 78% to 95%, close to the single-state. The only regime with relatively lower correlations is the rally, but the ergodic probabilities reveal its chances are only 21%. At the same time, equation (10) shows that the CGL model inverts the var-cov matrix to compute the weights. Then, high correlations lead to extremely low numbers, more sensitive to return variations.

Thus, more pronounced under less uncertain (volatile) states, high correlations might oversize portfolio hedges relative to the investor's risk aversion. Therefore, we propose to control leverage to contain it.

Table 2 – Estimated parameters

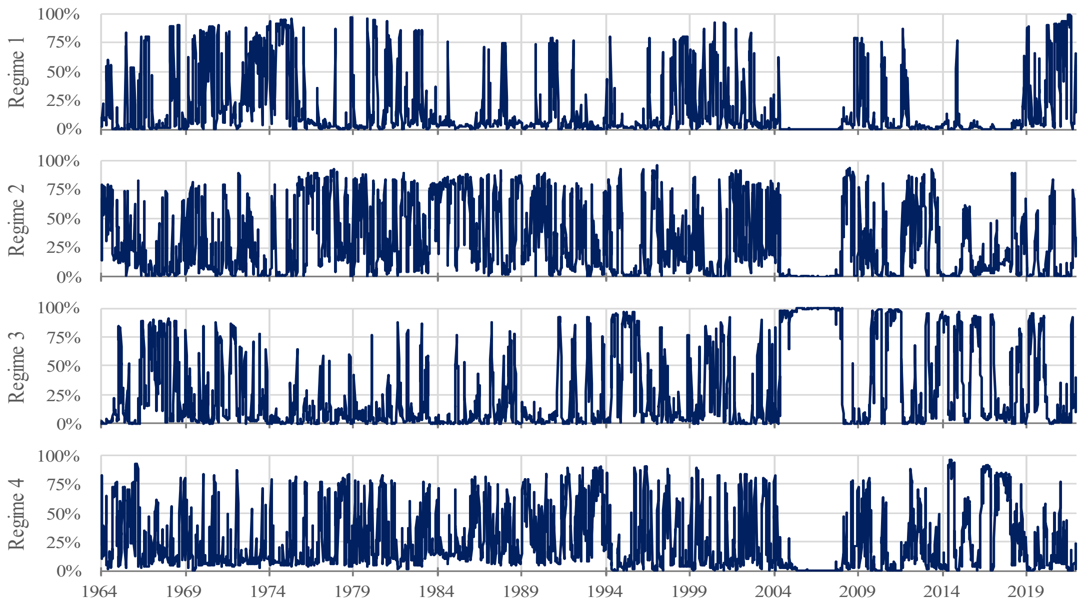
Panel A: Single State Model (%)			**Small Caps**	**Mid-Caps**	**Large Caps**
Expected Returns			9.57	9.67	7.45
Volatility and correlation matrix		Small Caps	18.40		
		Mid-Caps	92.57	17.91	
		Large Caps	76.10	88.30	15.45
Panel B.1: Four State Model (%)			**Small Caps**	**Mid-Caps**	**Large Caps**
Expected Returns		Regime 1 (crash)	-11.71	-11.16	-8.58
		Regime 2 (bear)	-12.52	-5.04	0.58
		Regime 3 (bull)	12.15	13.94	11.99
		Regime 4 (rally)	80.54	56.19	31.06
Volatility and correlation matrix	Regime 1 (crash)	Small Caps	32.12		
		Mid-Caps	93.02	31.66	
		Large Caps	78.78	90.09	26.86
	Regime 2 (bear market)	Small Caps	8.61		
		Mid-Caps	92.29	10.87	
		Large Caps	78.39	89.18	11.68
	Regime 3 (bull market)	Small Caps	17.27		
		Mid-Caps	94.36	15.11	
		Large Caps	82.55	90.42	10.92
	Regime 4 (rally)	Small Caps	7.41		
		Mid-Caps	83.32	8.22	
		Large Caps	54.92	75.41	9.81

Panel B.2: Four State Model (%)		**Regime 1**	**Regime 2**	**Regime 3**	**Regime 4**
Transition probabilities	Regime 1 (crash)	89.73	3.71	3.41	3.15
	Regime 2 (bear)	3.53	89.54	0.12	6.80
	Regime 3 (bull)	2.32	0.21	96.43	1.04
	Regime 4 (rally)	0.29	12.32	1.95	85.44
Steady state	Ergodic Prob.	18.11	31.46	29.69	20.74
	Duration (weeks)	10	10	28	7

Notes. This table was computed with excess returns over the risk-free rate. Correlation matrices present the volatilities in their diagonals. Weekly returns and volatilities are annualized for presentation. The indicated parameters were estimated considering the complete data set.

Figure 2 presents the out-of-sample probabilities, filtered from $t+1$ in t. For example, they show that crash state probabilities hit the highest level since 1964 during the COVID-19 pandemic, differently from the previous economic crisis. In addition, the probabilities reveal that before the subprime crisis, between 2004 and 2009, it was an unusually long bull market. Again, it contrasts with the other periods, where regimes frequently shift. The regime-switching models generate value by quantitatively tracking such shifts between economic cycles (states).

Figure 2 – Out-of-sample probabilities



Notes. The out-of-sample probabilities are the filtered probabilities from $t+1$ in t, estimated considering the rolling windows described in section 3.4.3.

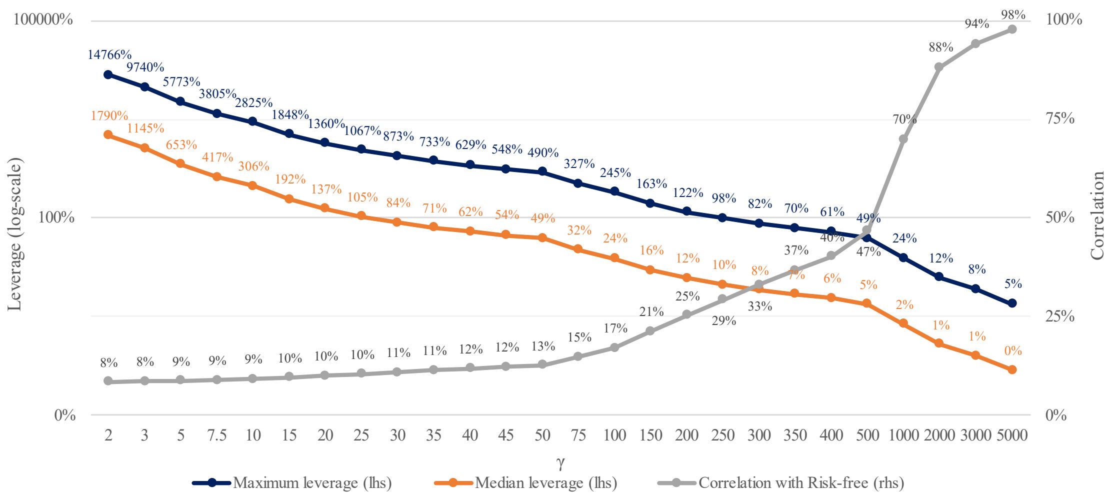
3.5.2 THE REGIME-SWITCHING BENCHMARKS

The unconstrained CGL model. It is the original configuration from Campani, Garcia, and Lewin (2021), i.e., it is the CGL model using fixed values of γ. Figure 3 shows that increasing γ conserves the significant distance between the leverage peaks and their median levels, despite mitigating the overall portfolio leverage. For example: to hold the maximum leverage below 100% in such a sample, the unconstrained CGL model requires $\gamma > 250$. Consequently, the median leverage drops below 10%, and the correlation between the portfolio and the risk-free returns rises above 29%. The correlations in Figure 3 indicate an increasing overweight of the risk-free asset when γ increases. In contrast, the literature usually represents the investor's risk preferences with

γ closer to 5, conducting the portfolio returns toward much lower correlations with the risk-free returns. For example, models with $\gamma \leq 25$ show correlations not greater than 10%, regardless of the leverage peaks greater than 10-fold the investor capital – which pose limitations either due to implementation or compliance rules.

To assess the tradeoff between the leverage levels and the risky assets allocation, at the out-of-sample exercise, we pair the unconstrained CGL with $\gamma = 75, 100, 200$ to the constrained portfolios at 100%, 200%, and 300%, respectively. However, Figure 3 reveals that it is inviable to obtain an unleveraged portfolio (leverage = 0%) only increasing the fixed value of γ, without achieving almost 100% correlation to the risk-free returns. Consequently, the comparison of this particular case will not carry an unconstrained regime-switching benchmark.

Figure 3 – The unconstrained CGL model

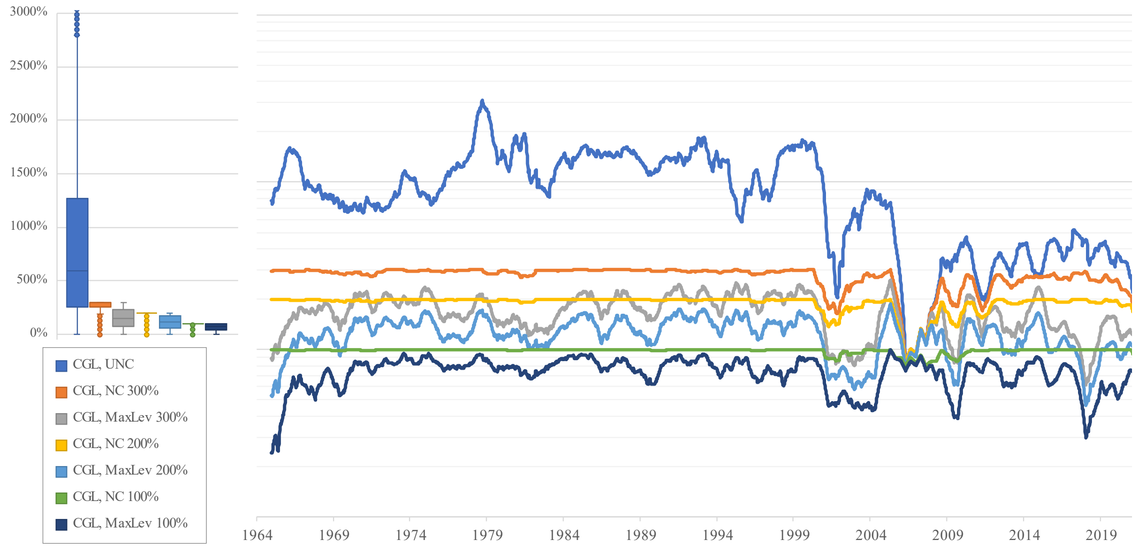


Notes. The unconstrained CGL model is the original model presented by Campani, Garcia, and Lewin (2021). The figure indicates the maximum and median portfolio leverage from the portfolios obtained considering such a model, and the correlation between the portfolios' returns and the risk-free returns, calculated in the out-of-sample exercise.

The numerically constrained CGL model. As presented in section 3.4, our base case is $\gamma = 5$; and as Figure 3 indicates, the median leverage of the unconstrained CGL of the base case is 653%. The left panel from Figure 4 additionally shows that such a portfolio leverages 25th and 75th percentiles are 306% and 1327%, respectively. This wide range results from the states dynamically shifting and resetting leverage according to the states' expectations. Figure 4 (left panel) also reveals that the leverage obtained by the numerically constrained CGL is flat at the boundary level, except for outliers. In other words, up to the leverage limit of 300%, the leverage from the numerically constrained CGL model virtually does not detach from the limit.

The right panel of Figure 4 also indicates that numerical constraints level the leverage of portfolios. Nevertheless, in cutting off such dynamics, they overlook the opportunity to mitigate risks by calibrating leverage according to regimes. In contrast, Figure 4, in both panels, shows that the CGL MaxLev model maintains the dynamic adjustments of the portfolio leverage while constraining the maximum leverage. Thus, the numerically constrained CGL model is expected to offer higher but more volatile returns than the CGL MaxLev.

Figure 4 – Portfolio leverage



Notes. The left panel shows the leverage distribution (scale limited to 3000%), while the right panel presents the 52-week moving average from the portfolios leverage calculated in the out-of-sample exercise. Both panels present the unconstrained CGL model (UNC), the numerically leverage CGL model (NC), and CGL MaxLev. The color key at the bottom fits both panels. All models were computed with $\gamma = 5$.

3.5.3 THE OUT-OF-SAMPLE PERFORMANCE

This section presents the results from the out-of-sample exercise performed from 1964 to 2021. As an additional robustness check, the supplementary file presents the results from its most recent subset, 2000 to 2021. Despite natural differences among the samples, the conclusions drawn from comparing the model's results are not significantly different.

Table 3 indicates that an investment in the risky assets admits volatilities up to 19% p-a. We will refer to it as an admissible market volatility. The risk of extreme events in these returns is indicated the by maximum drawdowns between 53% to 71%.

Table 3 – Risky assets and the risk-free rate

Assets (%)	**Small caps**	**Mid-caps**	**Large-caps**	**Risk-free**
Returns (p-a)	11.9	12.2	10.3	4.4
Volatility (p-a)	18.8	18.4	15.8	0.4
Sharpe ratio	40.2	42.8	37.5	-
Skewness	-0.7	-0.5	-0.4	0.6
Kurtosis	6.8	6.1	5.4	0.5
Maximum drawdown	71.3	53.6	53.0	0.0

Notes. This table presents the annualized results obtained from weekly observations from January 3rd, 1964 to November 30th, 2021.

The risky assets results are an intuitive benchmark, particularly for strategies without transaction costs. Additionally, we set the leverage constraints from the exercise to align the proposed strategies (volatility) with the market volatility. Therefore, we present constrained portfolios up to 300% leverage, as CGL MaxLev 300% volatility approximates to such a level.

Table 4 shows the out-of-sample results of the $1/n$ portfolio, SR model, and CGL models. The CGL models are presented unconstrained (UNC), numerically constrained (NC), and MaxLev. Below Table 4, we discuss the models indicated on its panels.

Table 4 – Out-of-sample results

A. Panel (%)		**Equal weights ($1/n$)**				**Single regime (SR)**				**CGL, γ = 5, UNC**			
A.I. Percentiles		**25**	**50**	**75**	**100**	**25**	**50**	**75**	**100**	**25**	**50**	**75**	**100**
Leverage	-	0.00	0.00	0.00	0.00	0.68	29.22	69.56	305.23	305.57	652.77	1326.50	5772.60
Weekly turnover	~LoT	0.21	0.34	0.51	3.70	0.91	1.56	2.82	25.23	9.85	31.70	77.06	1156.71
	LoT	-	-	-	-	-	-	-	-	0.00	0.00	63.27	941.99
A.II. Transaction Costs		**0.00**	**0.10**	**0.20**	**0.40**	**0.00**	**0.10**	**0.20**	**0.40**	**0.00**	**0.10**	**0.20**	**0.40**
Returns (p-a)	~LoT	10.10	8.98	8.96	8.91	14.90	13.54	13.32	12.89	546.76	385.15	267.01	109.05
	LoT	-	-	-	-	-	-	-	-	445.36	342.45	262.27	142.30
Volatility (p-a)	~LoT	12.59	12.59	12.59	12.59	19.09	19.08	19.08	19.08	99.91	99.50	99.39	100.03
	LoT	-	-	-	-	-	-	-	-	98.79	99.21	99.97	102.41
Sharpe ratio	~LoT	45.35	36.51	36.32	35.94	55.07	47.97	46.82	44.53	542.86	382.69	264.24	104.63
	LoT	-	-	-	-	-	-	-	-	446.39	340.75	257.96	134.66
Skewness	~LoT	-0.64	-0.64	-0.64	-0.64	-0.68	-0.69	-0.70	-0.72	2.48	2.22	1.94	1.32
	LoT	-	-	-	-	-	-	-	-	1.95	1.71	1.45	0.93
Kurtosis	~LoT	6.33	6.33	6.34	6.34	7.22	7.25	7.28	7.33	27.54	25.82	24.03	20.51
	LoT	-	-	-	-	-	-	-	-	33.04	30.78	28.46	23.92
Max. drawdown	~LoT	45.58	46.50	46.53	46.60	81.96	83.03	83.39	84.10	99.99	99.99	99.99	99.99
	LoT	-	-	-	-	-	-	-	-	99.99	99.99	99.99	99.99
B. Panel (%)		**CGL, γ = 10, UNC**				**CGL, γ = 25, UNC**				**CGL, γ = 50, UNC**			
B.I. Percentiles		**25**	**50**	**75**	**100**	**25**	**50**	**75**	**100**	**25**	**50**	**75**	**100**
Leverage	-	129.36	306.27	633.86	2824.96	46.84	104.95	227.48	1067.35	21.97	48.87	103.67	490.23
Weekly turnover	~LoT	7.74	25.56	66.75	801.01	5.10	18.24	51.06	543.61	3.30	13.23	35.56	336.68
	LoT	0.00	0.00	48.87	801.01	0.00	0.00	33.79	543.61	0.00	0.00	22.87	336.68
B.II. Transaction Costs		**0.00**	**0.10**	**0.20**	**0.40**	**0.00**	**0.10**	**0.20**	**0.40**	**0.00**	**0.10**	**0.20**	**0.40**
Returns (p-a)	~LoT	131.05	100.39	75.35	33.80	46.96	38.36	31.55	18.85	24.38	20.13	17.18	11.48
	LoT	121.40	99.16	80.77	48.47	44.91	38.19	33.08	23.36	23.66	20.14	17.88	13.48
Volatility (p-a)	~LoT	49.16	49.46	49.88	51.09	19.51	19.72	19.97	20.61	9.75	9.86	10.00	10.34
	LoT	47.42	48.00	48.73	50.55	19.07	19.36	19.70	20.51	9.71	9.85	10.02	10.43
Sharpe ratio	~LoT	257.62	194.10	142.25	57.57	218.23	172.32	135.98	70.16	205.02	159.56	127.87	68.59
	LoT	246.77	197.42	156.74	87.21	212.43	174.60	145.65	92.48	198.43	159.85	134.63	87.16
Skewness	~LoT	2.44	2.20	1.95	1.41	2.42	2.22	2.00	1.52	2.41	2.22	2.01	1.56
	LoT	2.66	2.47	2.26	1.79	2.45	2.27	2.07	1.62	2.35	2.16	1.96	1.51
Kurtosis	~LoT	26.83	25.72	24.53	22.05	26.36	25.54	24.61	22.61	26.03	25.32	24.50	22.68
	LoT	27.42	26.38	25.19	22.61	27.90	26.84	25.64	23.01	28.58	27.63	26.53	24.07
Max. drawdown	~LoT	58.31	71.41	81.03	94.77	26.77	31.79	41.40	64.56	13.95	15.28	20.28	38.59
	LoT	46.58	60.51	72.22	86.49	24.03	26.27	35.50	50.75	13.85	14.15	17.02	27.78
C. Panel (%)		**CGL, γ = 75, UNC**				**CGL, γ = 5, NC 300%**				**CGL, γ = 5, MaxLev 300%**			
C.I. Percentiles		**25**	**50**	**75**	**100**	**25**	**50**	**75**	**100**	**25**	**50**	**75**	**100**

Leverage	-	14.46	32.28	66.82	326.60	300.00	300.00	300.00	300.00	92.37	161.47	236.96	300.00
Weekly turnover	~LoT	2.50	9.98	27.26	231.55	4.07	11.60	43.69	335.83	6.40	20.54	50.65	342.65
	LoT	0.00	0.00	17.90	231.55	0.00	0.00	31.47	326.07	0.00	0.00	39.61	342.65
C.II. Transaction Costs		**0.00**	**0.10**	**0.20**	**0.40**	**0.00**	**0.10**	**0.20**	**0.40**	**0.00**	**0.10**	**0.20**	**0.40**
Returns (p-a)	~LoT	17.43	14.37	12.50	8.84	59.37	47.47	37.80	20.21	40.83	33.38	27.58	16.69
	LoT	16.94	14.31	12.87	10.02	57.93	47.95	39.95	25.13	38.59	32.76	28.44	20.18
Volatility (p-a)	~LoT	6.51	6.59	6.68	6.91	28.70	29.10	29.56	30.62	18.57	18.77	19.00	19.52
	LoT	6.53	6.63	6.74	7.01	28.04	28.48	28.97	30.12	18.34	18.58	18.86	19.49
Sharpe ratio	~LoT	200.28	151.48	121.37	64.44	191.56	148.05	113.02	51.67	196.30	154.48	122.07	63.01
	LoT	192.09	149.80	125.83	80.37	190.90	152.98	122.76	68.87	186.43	152.67	127.55	81.05
Skewness	~LoT	2.40	2.22	2.02	1.58	0.30	0.23	0.16	0.00	0.99	0.92	0.84	0.68
	LoT	2.25	2.07	1.86	1.43	0.15	0.09	0.02	-0.13	0.86	0.79	0.71	0.54
Kurtosis	~LoT	25.74	25.08	24.32	22.59	5.65	5.45	5.24	4.80	10.82	10.46	10.08	9.31
	LoT	28.84	27.95	26.91	24.54	5.40	5.23	5.04	4.65	10.27	9.95	9.60	8.86
Max. drawdown	~LoT	9.39	11.40	15.73	26.88	53.59	61.51	67.66	87.58	34.26	39.74	45.74	69.37
	LoT	9.71	10.51	13.16	18.55	54.64	59.85	65.67	76.66	31.94	35.31	37.28	46.01
D. Panel (%)		**CGL, γ = 100, UNC**				**CGL, γ = 5, NC 200%**				**CGL, γ = 5, MaxLev 200%**			
D.I. Percentiles		**25**	**50**	**75**	**100**	**25**	**50**	**75**	**100**	**25**	**50**	**75**	**100**
Leverage	-	10.65	24.03	49.94	244.87	200.00	200.00	200.00	200.00	70.09	123.54	180.11	200.00
Weekly turnover	~LoT	2.02	8.06	22.26	176.42	3.43	10.47	39.27	253.44	5.19	17.96	44.75	256.14
	LoT	0.00	0.00	14.23	180.36	0.00	0.00	26.38	253.44	0.00	0.00	34.09	256.14
D.II. Transaction Costs		**0.00**	**0.10**	**0.20**	**0.40**	**0.00**	**0.10**	**0.20**	**0.40**	**0.00**	**0.10**	**0.20**	**0.40**
Returns (p-a)	~LoT	14.07	11.55	10.18	7.49	44.55	36.22	29.64	17.37	31.87	26.26	22.10	14.16
	LoT	13.73	11.54	10.48	8.39	43.81	36.74	31.30	21.00	30.32	25.83	22.70	16.66
Volatility (p-a)	~LoT	4.90	4.96	5.02	5.19	23.05	23.33	23.65	24.40	14.41	14.58	14.76	15.17
	LoT	4.93	5.00	5.09	5.29	22.67	22.96	23.30	24.09	14.22	14.40	14.61	15.09
Sharpe ratio	~LoT	197.55	144.57	115.34	59.76	174.21	136.44	106.77	53.21	190.64	150.07	120.00	64.39
	LoT	189.33	142.87	119.75	75.69	173.89	140.88	115.49	68.97	182.41	148.90	125.37	81.33
Skewness	~LoT	2.40	2.22	2.02	1.59	0.09	0.04	-0.01	-0.13	0.77	0.72	0.66	0.52
	LoT	2.21	2.03	1.83	1.41	0.01	-0.04	-0.09	-0.19	0.53	0.47	0.41	0.27
Kurtosis	~LoT	25.42	24.81	24.09	22.44	4.59	4.46	4.32	4.02	9.69	9.36	9.01	8.31
	LoT	28.80	27.98	26.98	24.70	4.63	4.52	4.39	4.09	8.71	8.43	8.14	7.53
Max. drawdown	~LoT	7.05	9.94	13.46	20.56	48.62	52.06	55.73	70.68	30.11	33.71	35.82	52.49
	LoT	7.46	9.47	11.41	15.48	49.76	52.60	54.97	62.35	29.36	32.46	34.14	37.37
E. Panel (%)		**CGL, γ = 200, UNC**				**CGL, γ = 5, NC 100%**				**CGL, γ = 5, MaxLev 100%**			
E.I. Percentiles		**25**	**50**	**75**	**100**	**25**	**50**	**75**	**100**	**25**	**50**	**75**	**100**
Leverage	-	5.32	12.01	24.97	122.38	100.00	100.00	100.00	100.00	49.90	83.84	100.00	100.00
Weekly turnover	~LoT	1.13	4.60	12.52	106.83	2.56	8.79	34.35	175.36	3.32	12.70	35.06	188.69
	LoT	0.00	0.00	8.33	106.83	0.00	0.00	20.95	175.36	0.00	0.00	24.80	188.69

E.II. Transaction Costs		0.00	0.10	0.20	0.40	0.00	0.10	0.20	0.40	0.00	0.10	0.20	0.40
Returns (p-a)	~LoT	9.15	7.41	6.75	5.44	30.61	25.52	21.84	14.76	22.78	19.00	16.49	11.61
	LoT	8.99	7.39	6.88	5.86	30.51	26.09	23.04	17.14	22.48	19.31	17.37	13.59
Volatility (p-a)	~LoT	2.49	2.52	2.55	2.63	17.63	17.77	17.94	18.32	10.18	10.29	10.40	10.67
	LoT	2.52	2.55	2.59	2.68	17.36	17.51	17.68	18.09	10.04	10.15	10.28	10.59
Sharpe ratio	~LoT	190.98	119.80	92.46	39.93	148.76	118.93	97.26	56.60	180.63	142.05	116.29	67.66
	LoT	182.57	117.80	96.24	54.86	150.41	123.94	105.47	70.49	180.22	146.90	126.23	86.88
Skewness	~LoT	2.33	2.17	1.99	1.60	-0.21	-0.23	-0.26	-0.31	0.59	0.55	0.50	0.40
	LoT	2.08	1.91	1.73	1.33	-0.18	-0.21	-0.24	-0.30	0.33	0.29	0.24	0.14
Kurtosis	~LoT	23.77	23.32	22.75	21.39	4.29	4.21	4.12	3.92	9.40	9.11	8.80	8.16
	LoT	27.72	27.11	26.33	24.44	4.25	4.19	4.12	3.95	9.00	8.76	8.49	7.91
Max. drawdown	~LoT	3.46	8.17	9.99	13.86	42.51	45.44	47.86	52.75	25.56	25.79	26.87	32.12
	LoT	3.84	7.27	8.21	10.31	43.91	46.67	48.60	52.26	25.49	25.71	25.99	28.38

F. Panel (%)						CGL, γ = 5, NC 0%				CGL, γ = 5, MaxLev 0%			
F.I. Percentiles						25	50	75	100	25	50	75	100
Leverage	-					0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Weekly turnover	~LoT					0.63	4.42	20.23	100.00	1.31	5.38	16.93	100.00
	LoT					0.00	0.00	8.97	100.00	0.00	0.00	10.08	100.00
F.II. Transaction Costs						0.00	0.10	0.20	0.40	0.00	0.10	0.20	0.40
Returns (p-a)	~LoT					17.06	14.81	13.73	11.59	13.52	11.62	10.85	9.33
	LoT					17.14	15.07	14.16	12.36	13.20	11.47	10.87	9.68
Volatility (p-a)	~LoT					14.33	14.35	14.38	14.45	8.01	8.02	8.04	8.09
	LoT					14.17	14.20	14.23	14.30	7.87	7.89	7.92	7.97
Sharpe ratio	~LoT					88.45	72.61	64.92	49.80	114.01	90.13	80.34	61.06
	LoT					90.00	75.23	68.68	55.78	111.83	89.73	81.90	66.42
Skewness	~LoT					-0.61	-0.62	-0.62	-0.62	-0.10	-0.11	-0.12	-0.14
	LoT					-0.52	-0.53	-0.53	-0.54	0.01	0.00	-0.01	-0.03
Kurtosis	~LoT					4.56	4.54	4.52	4.46	7.14	7.11	7.08	7.01
	LoT					3.87	3.86	3.85	3.82	6.33	6.32	6.31	6.26
Max. drawdown	~LoT					40.22	42.28	43.28	46.15	19.94	21.60	22.18	23.31
	LoT					40.25	42.13	42.97	44.89	21.29	23.17	23.76	24.92

Notes. The table presents the out-of-sample results from the equal weights portfolio ($1/n$), the single regime (SR), and the CGL model. The SR model represents the allocation under the recursive utility function without multiple regimes (single state) with $\gamma = 5$. The CGL model is presented considering alternative configurations: unconstrained (UNC), numerically constrained (NC), and MaxLev. We present the unconstrained CGL with γ varying from 5 to 200. Meanwhile, the leverage constrains in NC and MaxLev are 0%, 100%, 200%, and 300%. The table is horizontally organized in panels, each with sections I and II. Section I presents the percentiles of distributions, as leverage and turnover are independent from the transaction costs. Section II is organized according to the transaction costs: 0.00%, 0.10%, 0.20%, and 0.40%. Except for the leverage data, the measures are segmented between ~LoT and LoT in both sections, indicating the results without and with LoT, respectively. The out-of-sample exercise was conducted with weekly observations from January 3rd, 1964 to November 30th, 2021.

Reference models (panel A). The $1/n$ portfolio is an unleveraged portfolio with the lowest turnover. It presents a Sharpe ratio of 36%, in line with the risky assets. Compared to them, the $1/n$ allocation lowers the maximum drawdown to approximately 46%, establishing an important benchmark. In turn, The SR model presents maximum leverage of 305%, while its turnover's 75th percentile is below 3%. Such a low turnover makes the costs in SR returns not as critical as in regime-switching models. Compared to the $1/n$ portfolio, the SR model has a slightly higher Sharpe ratio, but its maximum drawdown is almost two-fold. Despite the severe drawdown, highly leveraged portfolios eventually recover in long samples, as the case of the unconstrained CGL model with $\gamma = 5$ described below.

The unconstrained CGL model with $\gamma = 5$ is our base case, as it is in Campani, Garcia, and Lewin (2021). Notwithstanding its extreme leverage (analyzed in previous sections), the base case exposes the favorable impact of the LoT filter. The median turnover of 32% from the base case without LoT (optimal model) falls to 0% with LoT. The returns-to-risk measure (Sharpe ratio) reveals that LoT has a positive impact in this model when transaction costs are 0.40%. Moreover, the base case dominates $1/n$ portfolio and SR model in terms of return-to-risk, indicating the relevance of the regime-switching in allocation models considering the stochastic recursive utility function. However, due to the regime shifting process, the base case returns are more leptokurtic (thus, riskier) than the single-state benchmarks.

Unconstrained models (panel B). If, on the one hand, the CGL model with $\gamma = 10, 25, 50$ mitigates the leverage levels relative to the base case ($\gamma = 5$), on the other, these unconstrained configurations still lead to almost impractical leverage peaks: 2825%, 1067%, and 490%, respectively. The supplementary file presents the unconstrained CGL for more risk aversion settings. Still, one could also argue the outcome of applying MaxLev with $\gamma > 5$. We answer it by pointing out that the return-to-risk measures from the base case dominate those from the unconstrained models with γ > 5. As we learned from Figure 3, increasing γ amplifies the overweight in the risk-free asset. Thus, the dominating return-to-risk measures reveal that the CGL strategy has more chances of successful performance than failure, and those chances are unchanged using MaxLev. Hence, the following panels apply NC and MaxLev with $\gamma = 5$.

Moreover, panel B shows that LoT eliminates the turnover up to the 50th percentile. The return-to-risk measures from three models reveal that LoT is favorable in any case of transaction costs. Contrasting to the base case, it indicates that under less extreme leverage levels, LoT generates value. For example, take the unconstrained CGL model with $\gamma = 25$. The Sharpe ratios for transaction costs of 0.20% are 134% and 146% for the cases without and with LoT filter, respectively. Naturally, these ratios decrease while increasing costs. Nevertheless, for transaction costs of 0.40%, the Sharpe ratio without LoT is 70%, while with LoT it is 92%. This pattern repeats over the other models, evidencing that LoT creates value when allocating for transaction costs, most notably when costs increase.

Maximum leverage at 300% (panel C). The maximum leverage of the unconstrained CGL model with $\gamma = 75$ is 327%, yet we use it to benchmark the cases where leverage is limited to 300%. The leverage from the unconstrained CGL model with $\gamma = 100$ is 245%. Although a fine-tuning between $75 < \gamma < 100$ would result in maximum leverage closer to 300%, it is not critical to the current comparison. On the other hand, it is relevant to emphasize that these leverage levels correspond to the entire sample (1964-2021). The supplementary file shows the subset 2000-2021, where the maximum leverage for the same risk preference is significantly different. The difference between the maximum leverage magnitudes results from estimating more (or less) uncertain economic states in the samples. Hence, only increasing γ is insufficient to effectively control (extreme) leverage at discretionary levels.

Sill, given an unconstrained case, $\gamma = 75$ benchmarks the return-to-risk measures relative to the limit of 300% leverage. Panel C shows that those measures from CGL MaxLev 300% are aligned with the unconstrained case. Meanwhile, we cannot lose sight that these unconstrained returns are more exposed to the risk-free rate (Figure 3). Thus, matching Sharpe ratios reflect that MaxLev manages risks more effectively than the unconstrained model.

At the same time, considering transaction costs, the Sharpe ratios from CGL MaxLev 300% dominate those from CGL NC 300%. It results from an overly leveraged base case, where the NC procedure flattens leverage at the limits (Figure 4); and while higher leverage levels increase returns, they also increase volatility. The outperformance from CGL MaxLev 300% relative to NC 300% is even magnified when considering higher transaction costs, underlining that it is valuable to dynamically adjust the leverage levels upon uncertainty. So, at the level of 300% leverage, the higher the costs, the more positive it is to constrain leverage using MaxLev.

Maximum leverage at 200% (panel D). The conclusions from comparing returns-to-risk are analogous to those from panel C. First, the alignment between Sharpe ratios with CGL unconstrained model ($\gamma = 100$) indicates that CGL MaxLev 200% offers superior risky assets management than such a benchmark, given a lower risk-free allocation. Then, at the same cost level, MaxLev's return-to-risk dominating those from NC evidence that MaxLev is a more effective constraining procedure at 200% leverage.

Another characteristic from MaxLev is that it skews the distribution slightly to the right side relative to the NC model. It results from MaxLev mitigating leverage proportionally between regimes. Then, when MaxLev adjusts leverage upon regime expectations, it mitigates the risk of negative events. In contrast, when leveling leverage by its limit, the NC procedure does not balance leverage between regimes. Thus, compared to NC returns, MaxLev's are consistently slightly more skewed to the right, pointing to a greater number of positive returns.

Maximum leverage at 100% (panel E). In contrast with panels C and D, panel E shows that CGL MaxLev 100% considering transaction costs outperforms its benchmarks' Sharpe ratio more pronouncedly. Therefore, it suggests that MaxLev generates even greater value for lower leverage limits. In addition, as in previous panels, MaxLev significantly lowers maximum drawdowns relative to NC, suggesting that it mitigates left-tail events in comparison to NC procedures. These findings are extensible to the comparison with $1/n$ portfolio and SR model.

Unleveraged cases (panel F). Constraining the portfolios at a 0% leverage level represents the unleveraged condition. It is a special case, as some investors cannot hold any leveraged positions. Hence, we suppress an unconstrained model from panel F, as it would require extremely high values of γ for the unconstrained CGL to achieve the unleveraged condition. Furthermore, such a case would present negative risk premia considering transaction costs (the supplementary file shows unconstrained models approximating to 0% leverage). In contrast, the unleveraged condition is achieved by the $1/n$ portfolio along with both models from panel F. Comparing them using the same costs, the return-to-risk measures from the CGL MaxLev 0% dominate the benchmarks. At the same time, it also presents the lowest maximum drawdown amongst them. Such results evidence that MaxLev effectively generates unconstrained portfolios without eroding the return-to-risk measures.

3.5.4 ROBUSTNESS CHECKS

Table 5 – Certainty equivalent returns differences ($\Delta CER, \%$)

TC (%)	Single regime (SR)	Equal weights (1/n)	CGL, MaxLev, ~LoT	CGL, UNC, ~LoT	CGL, UNC, LoT	CGL, NC, ~LoT	CGL, NC, LoT
MaxLev 300%, LoT							
0.00	2.10 [1.26 – 3.17]	2.63 [1.81 – 3.62]	-0.11 [-0.22 – 0.01]	1.21 [0.94 – 1.49]	1.32 [1.05 – 1.58]	-0.13 [-0.41 – 0.16]	-0.20 [-0.46 – 0.09]
0.10	1.92 [1.01 – 3.08]	2.50 [1.62 – 3.62]	-0.02 [-0.15 – 0.12]	1.39 [1.08 – 1.68]	1.44 [1.14 – 1.73]	-0.01 [-0.33 – 0.34]	-0.19 [-0.48 – 0.14]
0.20	1.61 [0.66 – 2.8]	2.15 [1.23 – 3.28]	0.11 [-0.05 – 0.26]	1.35 [1.00 – 1.67]	1.33 [0.98 – 1.65]	1.08 [0.53 – 1.88]	0.52 [0.09 – 1.09]
0.40	0.82 [-0.26 – 2.11]	1.28 [0.23 – 2.47]	0.52 [0.27 – 0.84]	1.24 [0.73 – 1.66]	1.02 [0.50 – 1.44]	0.19 [-0.28 – 0.86]	-0.37 [-0.76 – 0.10]
MaxLev 200%, LoT							
0.00	1.89 [1.05 – 2.94]	2.41 [1.60 – 3.39]	-0.07 [-0.19 – 0.04]	1.02 [0.72 – 1.33]	1.13 [0.82 – 1.41]	-0.01 [-0.32 – 0.32]	-0.07 [-0.37 – 0.24]
0.10	1.70 [0.79 – 2.86]	2.28 [1.40 – 3.4]	0.01 [-0.13 – 0.14]	1.23 [0.89 – 1.55]	1.30 [0.96 – 1.6]	0.06 [-0.29 – 0.45]	-0.10 [-0.42 – 0.26]
0.20	1.40 [0.47 – 2.6]	1.95 [1.04 – 3.07]	0.12 [-0.03 – 0.27]	1.14 [0.76 – 1.49]	1.16 [0.78 – 1.49]	1.02 [0.48 – 1.80]	0.53 [0.08 – 1.09]
0.40	0.68 [-0.37 – 1.94]	1.13 [0.12 – 2.32]	0.49 [0.26 – 0.76]	0.90 [0.35 – 1.34]	0.78 [0.24 – 1.22]	0.20 [-0.3 – 0.87]	-0.30 [-0.72 – 0.19]
MaxLev 100%, LoT							
0.00	1.59 [0.78 – 2.62]	2.11 [1.34 – 3.07]	-0.01 [-0.12 – 0.10]	0.48 [0.15 – 0.81]	0.56 [0.23 – 0.87]	0.19 [-0.16 – 0.59]	0.11 [-0.22 – 0.48]
0.10	1.40 [0.54 – 2.53]	1.99 [1.15 – 3.07]	0.07 [-0.06 – 0.18]	0.74 [0.38 – 1.10]	0.80 [0.44 – 1.13]	0.18 [-0.21 – 0.63]	0.01 [-0.34 – 0.42]
0.20	1.16 [0.27 – 2.31]	1.70 [0.85 – 2.79]	0.16 [0.03 – 0.30]	0.62 [0.21 – 1.00]	0.65 [0.25 – 1.01]	0.88 [0.36 – 1.61]	0.47 [0.03 – 1.03]
0.40	0.57 [-0.39 – 1.78]	1.03 [0.11 – 2.14]	0.45 [0.27 – 0.67]	0.30 [0.24 – 0.76]	0.28 [0.26 – 0.70]	0.20 [-0.30 – 0.85]	-0.21 [-0.65 – 0.30]
MaxLev 0%, LoT							
0.00	0.66 [0.03 – 1.48]	1.18 [0.67 – 1.83]	-0.01 [-0.09 – 0.17]			0.33 [-0.03 – 0.73]	0.24 [-0.10 – 0.61]
0.10	0.51 [0.15 – 1.4]	1.09 [0.55 – 1.82]	0.02 [-0.07 – 0.20]			0.15 [-0.21 – 0.56]	0.03 [-0.31 – 0.40]
0.20	0.41 [0.26 – 1.31]	0.95 [0.41 – 1.67]	0.06 [0.03 – 0.25]			0.44 [0.02 – 0.97]	0.23 [-0.15 – 0.68]
0.40	0.18 [-0.5 – 1.11]	0.64 [0.12 – 1.34]	0.16 [0.05 – 0.36]			0.12 [-0.27 – 0.60]	-0.09 [-0.44 – 0.31]

Notes. The table indicates the differences between annualized certainty equivalent returns (ΔCER). The differences correspond to the CER from the models indicated in the horizontal panels, minus the CER from the benchmarks in the columns – computed according to section 3.4.5. The horizontal panels present CGL MaxLev 300%, 200%, 100%, and 0% with LoT, under the transaction costs (TC): 0.00%, 0.10%, 0.20%, and 0.40%. The benchmarks in the columns are the SR model, the $1/n$ portfolio, the CGL MaxLev model without LoT (~LoT), the unconstrains CGL model with and without LoT, and the numerically constrained CGL model (NC) with and without LoT. As in section 3.5.3, the models with $\gamma = 75, 100, 200$ are the benchmarks for CGL MaxLev 300%, 200%, and 100%, respectively. Section 3.5.3 also demonstrates that unconstrained CGL models do not offer unleveraged models. Furthermore, CGL MaxLev ~LoT, CGL NC ~LoT, and CGL NC LoT are paired with the horizontal panels at the correspondent leverage levels (300%, 200%, 100%, 0%). Below the ΔCER, we report the 95% bootstrap confidence intervals drawn from 1,000,000 samples with replacement, with the bias-corrected and accelerated percentile method. The out-of-sample exercise was conducted with weekly observations from January 3rd, 1964 to November 30th, 2021.

Table 5 shows the annualized certainty equivalent returns differences (ΔCER) between CGL MaxLev with LoT and the benchmarks. Below we analyze them by the benchmarks.

Single regime. Positive ΔCER and positive confidence intervals indicate that CGL MaxLev with LoT statistically outperforms the SR model, considering transaction costs up to 0.20%. Yet, for the highest transaction cost (0.40%), although ΔCER is positive, confidence intervals with different algebraic signals indicate that they are not statistically significant.

Equal weights. Positive ΔCER and positive confidence intervals indicate that CGL MaxLev with LoT surpass $1/n$ portfolios at any cost level with statistical significance.

CGL MaxLev without LoT. Table 5 confirms a conclusion from section 3.5.3: LoT generates value considering higher transaction costs and low leverage limits. With 200% and 300% leverage constraints, CGL MaxLev with LoT outperforms the correspondent model without LoT considering the cost of 0.40% with statistical significance. Yet, lowering the maximum leverage limit, LoT also statistically outperforms the benchmark at the cost of 0.20%.

Unconstrained models. Positive ΔCER and positive confidence intervals indicate that CGL MaxLev with LoT statistically outperforms the unconstrained CGL models at any level of transaction costs. However, to unleverage the unconstrained CGL model, it would be necessary to set γ at uncommonly high values, resulting in an overweighted risk-free allocation (Figure 3). Thus, as section 3.5.3, we suppress such a benchmark for the unleveraged condition.

Numeric constrained models. At the leveraged portfolios, CGL MaxLev with LoT statistically outperforms the NC models, considering transaction costs of 0.20%. In the remaining pairs, the values from ΔCER are not statistically different from zero, thus we cannot demonstrate that one model outperforms the other. However, given we already demonstrated that MaxLev outperforms the unconstrained model, the similarity to the NC models is not uninteresting. It indicates that MaxLev's lower volatility compensates NC models high returns, in the certainty equivalence – suggesting that MaxLev is a competitive constraining procedure.

3.6 CONCLUSION

We addressed the issue of constraining dynamic models for multiple regimes economies for recursive utility preferences applying the CGL model to solve an allocation strategy accounting for transaction costs – a gap left by Campani, Garcia, and Lewin (2021) and Lewin and Campani (2020a,b). We studied a single asset class portfolio, given by equities, to observe the effects of highly correlated assets. In this setting, we identified four unobservable regimes with the characteristics of crash, bear market, bull market, and rally states.

There is a body of literature indicating that regime-switching portfolio strategies are challenged by elevated leverage and turnover, but solutions by Bulla, Mergner, Bulla, Sesboüé, and Chesneau (2011), Fiecas, Franke, Von Sachs, and Tadjuidje (2017) or Dal Pra, Guidolin, Pedio, and Vasile (2018) are ineffective for our application. Thus, we propose filters to control the portfolio maximum leverage (MaxLev) and low-turnover (LoT), using the CGL model.

We conducted an out-of-sample exercise that indicated that the CGL model with MaxLev and LoT report competitive return-to-risk measures relative to both single-state and regime switching benchmarks. For example, comparing models with and without

LoT, such a turnover control generates value considering transaction costs (and most notably when costs increase), as expected. Meanwhile, the annualized certainty equivalent returns indicate that CGL MaxLev outperforms the unconstrained CGL model (i.e., the original model from Campani, Garcia, and Lewin, 2021) with statistical significance. Furthermore, compared to a conventional numerically constraining procedure, MaxLev mitigates the returns' volatility.

The scope of our study was to evaluate regime-switching portfolio strategies for recursive preferences for constrained applications. Future studies shall find research opportunities investigating the proposed filters for new settings such as portfolios with different correlations in the assets menu, and a more complex cost structure, like dynamically observed illiquidity costs, and/or additional types of filters as maximum drawdown control and stop-loss triggers.

3.7 REFERENCES

Ang, Andrew and Bekaert, Geert (2002). International Asset Allocation with Regime Shifts. *The Review of Financial Studies*, *15*(4), 1137–1187. https://doi.org/10.1093/rfs/15.4.1137

Ang, Andrew and Timmermann, Allan (2012). Regime Changes and Financial Markets. *Annual Review of Financial Economics*, *4*(1), 313–337. https://doi.org/10.1146/annurev-financial-110311-101808

Bulla, Jan; Mergner, Sascha; Bulla, Ingo; Sesboüé, André; and Chesneau, Christophe (2011). Markov-Switching Asset Allocation: Do Profitable Strategies Exist?. *Journal of Asset Management*, *12*(5), 310–321. https://doi.org/10.1057/jam.2010.27

Campani, Carlos Heitor and Garcia, René (2019). Approximate Analytical Solutions for Consumption/Investment Problems Under Recursive Utility and Finite Horizon. *The North American Journal of Economics and Finance*, *48*, 364–384. https://doi.org/10.1016/j.najef.2019.03.005

Campani, Carlos Heitor; Garcia, René; and Lewin, Marcelo (2021). Optimal Portfolio Strategies in the Presence of Regimes in Asset Returns. *Journal of Banking and Finance*, *123*(106030), 1–17. https://doi.org/10.1016/j.jbankfin.2020.106030

Çanakoğlu, Ethem and Özekici, Süleyman (2012). HARA Frontiers of Optimal Portfolios in Stochastic Markets. *European Journal of Operational Research*, *221*(1), 129–137. https://doi.org/10.1016/j.ejor.2011.10.012

Clarke, Roger; De Silva, Harindra; and Thorley, Steven (2002). Portfolio Constraints and the Fundamental Law of Active Management. *Financial Analysts Journal*, *58*(5), 48–66. https://doi.org/10.2469/faj.v58.n5.2468

Dal Pra, Giulia; Guidolin, Massimo; Pedio, Manuela; and Vasile, Fabiola (2018). Regime Shifts in Excess Stock Return Predictability: An Out-Of-Sample Portfolio Analysis. *The Journal of Portfolio Management*, *44*(3), 10–24. https://doi.org/10.3905/jpm.2018.2018.1.077

DeMiguel, Victor; Garlappi, Lorenzo; and Uppal, Raman (2009). Optimal Versus Naive Diversification: How Inefficient is the 1/N Portfolio Strategy? *The Review of Financial Studies*, *22*(5), 1915–1953. https://doi.org/10.1093/rfs/hhm075

DeMiguel, Victor; Garlappi, Lorenzo; Nogales, Francisco; and Uppal, Raman (2009). A Generalized Approach to Portfolio Optimization: Improving Performance by Constraining Portfolio Norms. *Management Science*, *55*(5), 798–812. https://doi.org/10.1287/mnsc.1080.0986

Duffie, Derrel and Epstein, Larry (1992). Stochastic Differential Utility. *The Econometric Society*, *60*(2), 353–394. https://doi.org/10.2307/2951600

Epstein, Larr, and Zin, Stanley (1989). Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework. *Econometrica*, *57*(4), 937–969. https://doi.org/10.1142/9789814417358_0012

Fiecas, Mark; Franke, Jürgen; Von Sachs, Rainer; and Tadjuidje, Joseph (2017). Shrinkage Estimation for Multivariate Hidden Markov Models. *Journal of the American Statistical Association*, *112*(517), 424–435. https://doi.org/10.1080/01621459.2016.1148608

Fugazza, Carolina; Guidolin, Massimo; and Nicodano, Giovanna (2015). Equally Weighted vs. Long-Run Optimal Portfolios. *European Financial Management*, *21*(4), 742–789. https://doi.org/10.1111/eufm.12042

Gârleanu, Nicolae and Pedersen, Lasse (2013). Dynamic Trading with Predictable Returns and Transaction Costs. *The Journal of Finance*, *68*(6), 2309–2340. https://doi.org/10.1111/jofi.12080

Graflund, Andreas and Nilsson, Birger (2003). Dynamic Portfolio Selection: The Relevance of Switching Regimes and Investment Horizon. *European Financial Management*, *9*(2), 179–200. https://doi.org/10.1111/1468-036X.00215

Guidolin, Massimo and Hyde, Stuart (2012). Can VAR Models Capture Regime Shifts in Asset Returns? A Long-Horizon Strategic Asset Allocation Perspective. *Journal of Banking and Finance*, *36*(3), 695–716. https://doi.org/10.1016/j.jbankfin.2011.10.011

Guidolin, Massimo and Ono, Sadayuki (2006). Are the Dynamic Linkages Between the Macroeconomy and Asset Prices Time-Varying? *Journal of Economics and Business*, *58*(5–6), 480–518. https://doi.org/10.1016/j.jeconbus.2006.06.009

Guidolin, Massimo and Timmermann, Allan (2007). Asset Allocation Under Multivariate Regime Switching. *Journal of Economic Dynamics and Control*, *31*(11), 3503–3544. https://doi.org/10.1016/j.jedc.2006.12.004

Guidolin, Massimo and Timmermann, Allan (2008). International Asset Allocation Under Regime Switching, Skew, and Kurtosis Preferences. *The Review of Financial Studies* 21(2), 889-935. https://doi.org/10.1093/rfs/hhn006

Hamilton, James. D. (1989). A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle. *The Econometric Society*, *57*(2), 357–384. https://doi.org/10.2307/1912559

Kraft, Holger; Seiferling, Thomas; and Seifried, Frank Thomas (2017). Optimal Consumption and Investment with Epstein–Zin Recursive Utility. *Finance and Stochastics*, *21*(1), 187–226. https://doi.org/10.1007/s00780-016-0316-0

Ledoit, Oliver and Wolf, Michael (2004). A Well-Conditioned Estimator for Large-Dimensional Covariance Matrices. *Journal of Multivariate Analysis*, *88*(2), 365–411. https://doi.org/10.1016/S0047-259X(03)00096-4

Lewin, Marcelo and Campani, Carlos Heitor (2020a). Portfolio Management Under Multiple Regimes: Strategies that Outperform the Market. *Revista de Administração Contemporânea*, *24*(4), 300–316. https://doi.org/10.1590/1982-7849rac2020190161

Lewin, Marcelo and Campani, Carlos Heitor (2020b). Portfolio Management Under Multiple Regimes: Out-of-sample Performance. *Brazilian Review of Finance*, *18*(3), 52. https://doi.org/10.12660/rbfin.v18n3.2020.81210

Liu, Hening (2011). Dynamic Portfolio Choice Under Ambiguity and Regime Switching Mean Returns. *Journal of Economic Dynamics and Control*, *35*(4), 623–640. https://doi.org/10.1016/j.jedc.2010.12.012

Lunde, Asger and Timmermann, Allan (2004). Duration Dependence in Stock Prices: An Analysis of Bull and Bear Markets. *Journal of Business and Economic Statistics*, *22*(3), 253–273. https://doi.org/10.1198/073500104000000136

Munk, Claus (2013). Financial Asset Pricing Theory. *Oxford University Press*,, 6th ed.. https://doi.org/10.1093/acprof:oso/9780199585496.001.0001

Nystrup, Peter; Boyd, Stephen; Lindström, Erik; and Madsen, Henrik (2019). Multi-Period Portfolio Selection with Drawdown Control. *Annals of Operations Research*, *282*(1–2), 245–271. https://doi.org/10.1007/s10479-018-2947-3

Sass, Jörn and Haussmann, Ulrich (2004). Optimizing the Terminal Wealth Under Partial Information: The Drift Process as a Continuous Time Markov Chain. *Finance and Stochastics*, *8*(4), 553–577. https://doi.org/10.1007/s00780-004-0132-9

Xing, Hao (2017). Consumption–Investment Optimization with Epstein–Zin Utility in Incomplete Markets. *Finance and Stochastics*, *21*(1), 227–262. https://doi.org/10.1007/s00780-016-0297-z

4. OPTIMAL CONSTRAINED STRATEGIES FOR FACTOR-BASED INVESTING IN THE BRAZILIAN STOCK MARKET[9]

4.1. ABSTRACT

We apply a regime-switching model with the stochastic recursive utility function to find optimal investment strategies for a factor-based stock portfolio in Brazil. Following Fama and French (1993) and Carhart (1997), we identify four risk factors in the Brazilian stock market. Then, we create investment strategies to diversify amongst them using the model from Campani, Garcia, and Lewin (2021). In an out-of-sample exercise, we benchmark those regime-switching strategies with single-state passive and active strategies. The Sharpe ratios from the regime-switching strategies outperform the single-state strategies under the complete sample and on shorter subsamples of the exercise. The samples also reveal that the unleveraged regime-switching strategy offers the lowest volatility. Finally, the certainty equivalent returns show that the regime-switching strategies outperform the benchmarks with statistical significance.

[9] This section presents the third article of the Thesis. We look forward to the comments of the examining board to submit it to an academic journal.

4.2 INTRODUCTION

Some critical questions for investors relate to the main drivers of stock returns and how to diversify among them. Sharpe (1964) and Lintner (1965), for instance, approached the first question by introducing the CAPM model, where different exposures to market risk (beta) describe the variations in the expected excess returns of stocks. Next, Fama and French (1993) presented a three-factor model based on the excess return of the market portfolio, the return of a portfolio long in small stocks and short in big stocks (small minus big, SMB), and the return of a portfolio long in high book-to-market stocks and short in low book-to-market stocks (high minus low, HML). They demonstrate that adding these two portfolios, often named size and value factors, as additional risk factors lead to a better explanation of the cross-section of average stock returns. Later, Carhart (1997) extended the three-factor model with a momentum factor, a portfolio long in winner stocks and short in loser stocks (momentum, MOM).

Although the literature presents additional risk factors for equities (Ang, 2014; Fama and French, 2015; and Hou et al., 2015), we set our scope under the classical Fama-French-Carhart factors, i.e., the four-factor model. We follow Chincoli and Guidolin (2017), who studied the four factors as main market drivers. Thus, we will refer to SMB, HML, and MOM as the investable portfolios that mimic the size, value, and momentum factors, as Ferson et al. (2006).

We address the second critical question concerning diversification strategies, pondering that a portfolio formed by different factors allows the investor to select between different types of risks. For instance, if one expects a factor to outperform the market, she can increase its exposure through a portfolio where such a factor's weight is greater than the market's. That is the transformation from a passive factor investment into an active strategy, says Ang (2014).

The literature offers a rich documentation of the shortcomings of passive investing. For example, Haghani and Dewey (2016) highlight the threats of holding a portfolio with weights predetermined by market values in the face of bubbles and market panic. Furthermore, Blitz (2020) documents that SMB and HML have individually experienced a negative average return in the US stocks over 2010-2019, just as MOM's returns have significantly receded compared to a more extended period, 1963-2019.

On the other hand, we find support for active strategies, first, in Barroso and Santa-Clara (2015) and Daniel and Moskowitz (2016), who argue that strategies that trail volatility could mitigate momentum crashes. Then, Chincoli and Guidolin (2017) state that factors can help to diversify beyond the traditional value-weighted approach, demonstrating the importance of estimating the conditional moments of the factor's returns. They indicate that regime-switching models provide superior estimates and more profitable portfolio strategies than other multivariate models. Similarly, Perez-Quiros and Timmermann (2000), Black and McMillan (2004), Guidolin and Timmermann (2008b), Tu (2010), and Gulen et al. (2011) indicate that factor-based portfolios built on regime-switching models outperform single-state benchmarks. However, none of these conclusions are based on recursive utility functions, considered more realistic for investment purposes. For example, Tu (2010) uses the quadratic utility while Chincoli and Guidolin (2017) work with the power utility function.

Recursive utility functions are ahead of those, as they hold the assumption that tomorrow’s felicity also depends on today’s felicity, and they are not time-additive. Moreover, they enable configuring investment preferences by timing the resolution of uncertainty, which impact the allocation model by the disentanglement of the relative risk aversion from consumption decisions over time (elasticity of intertemporal substitution). The function that captures such preferences in continuous time is the stochastic differential utility from Duffie and Epstein (1992).

The allocation strategy presented by Campani, Garcia, and Lewin (2021), i.e., the CGL model, is, so far, the only model using the stochastic differential recursive utility function in a regime-switching framework that presents a closed-form solution for the allocation problem. It is an approximate solution based on Campani and Garcia (2019). The authors demonstrate that it is sufficiently accurate. Before the CGL model, the literature solved dynamic allocation problems under regimes using power utility functions via numerical methods, such as the Monte Carlo simulation, as in Sass and Haussmann (2004), Guidolin and Timmermann (2007), and Liu (2011) or under very special conditions, such as in Wachter (2002), who provides analytical formulas to solve power utility problems but limited to a perfect negative correlation between the asset returns and the predictor variable; and Honda (2003), who finds a closed-form solution for the portfolio problem limited to the case of a constant relative risk aversion equal to 0.5.

Rouwenhorst (1999) presents evidence that emerging market stocks also exhibit SMB, HML, and MOM. He makes an international investigation with 1705 stocks from 20 emerging markets from every continent. In Americas, for instance, he studies firms from Argentina, Brazil, Chile, Colombia, Mexico, and Venezuela. Without proposing an allocation strategy, his research demonstrated that the factors that drive cross-sectional differences in expected stock returns in emerging equity markets are qualitatively similar to developed markets. In turn, Chague and Bueno (2008) and Santos et al. (2012) reveal that the three- and four-factor models are valid for Brazilian stocks, respectively. However, both applied passive allocation strategies. Meanwhile, Chen and Kawaguchi (2018) study a factor-based portfolio under a regimes' framework, using the quadratic utility to form strategies with Chinese stocks. Following the gap related to the developed markets, to our knowledge, there are no applications of a factor-based portfolio in emerging markets using an optimal allocation with recursive utility configured under hidden regimes on a Markov process.

The CGL model has never been applied under the perspective of factor-based investing, i.e., investors that form their portfolios using risk-factor portfolios or indices. Lewin and Campani (2020a) demonstrate the CGL model's accuracy with a portfolio of Brazilian equities, bonds, and international stocks impacted by the exchange rate. In an out-of-sample exercise, Lewin and Campani (2020b) show that the CGL model outperforms reference portfolios with equities, different bond classes, and the exchange rate. Meanwhile, Lewin and Campani (2022) apply the CGL model in the US stock market using different transaction cost levels. The latter authors also introduce a leverage control to allow constrained strategies in the CGL model framework, which affects regime-based portfolios, and will be addressed on the current study.

We thus investigate the performance of a factor-based strategy formed with a regime-switching model using such a state-of-the-art function, the recursive utility. Our regional choice was Brazil due to its representativeness among other emerging stock markets. Furthermore, it enabled concentrating on a single country, avoiding the direct impact of exchange rates over the regimes' estimation.

This study computes SMB, HML, and MOM portfolios from the Brazilian stock market at B3 (Brazilian exchange), then applies the constrained CGL model to estimate the strategies based on regimes for the factor-based investor. Finally, in an out-of-sample exercise with transaction costs, we compare the CGL performance to active and passive strategies interpreted as benchmarks. The results indicate that the Sharpe ratio from the

CGL model outperforms the benchmarks, and the certainty equivalent returns show that such outperformance is statistically significant.

4.3 METHODOLOGY

4.3.1 THE REGIME-SWITCHING ECONOMY

We assume an investor in a continuous-time model with a regime-switching economy governing assets returns. The investor maximizes her stochastic differential recursive utility function with an optimal portfolio allocation strategy.

State variable. Following Hamilton (1989), we consider an economy governed by the unobservable state variable Y_t describing an independent Markov chain process, given $R = \{1, 2, \dots, m\}$, where R is a finite set of m possible regimes. Then, we treat the state variable behavior by transition probabilities, which will rule if the economy remains at the same regime or jump to a new one after an exponentially distributed length of time, as following:

$$P_{ij,\Delta t} = \frac{\lambda_{ij}}{\sum_{k\neq i} \lambda_{ik}} \left(1 - e^{-\sum_{k\neq i} \lambda_{ik} \Delta t}\right), \text{ with } j \neq i \in R \text{ and } \lambda_{ij} > 0, \quad (1)$$

where $\lambda_{ii} = -\sum_{k\neq i} \lambda_{ik} \leq 0$ such that $P_{ij,\Delta t} = \frac{\lambda_{ij}}{-\lambda_{ii}}\left(1 - e^{\lambda_{ii}\Delta t}\right)$. The probability of staying at the same regime i over the next Δt is given by $P_{ii,\Delta t} = e^{\lambda_{ii}\Delta t}$, with λ_{ij} assumed constant.

Assets dynamics. The risk factors are represented by n risky assets, and the excess returns ($\hat{r}$) over the riskless asset (r_f) are defined through the multidimensional stochastic process:

$$\begin{bmatrix} d\hat{r}_{1,t} \\ d\hat{r}_{2,t} \\ \dots \\ d\hat{r}_{n,t} \end{bmatrix} = \boldsymbol{\mu_{s,t}}\, dt + \boldsymbol{\sigma_{s,t}}\, \boldsymbol{dZ_t} = \begin{bmatrix} \mu_{1,t} \\ \mu_{2,t} \\ \dots \\ \mu_{n,t} \end{bmatrix} dt + \begin{bmatrix} \sigma_{11,t} & 0 & \dots & 0 \\ \sigma_{21,t} & \sigma_{22,t} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \sigma_{n1,t} & \sigma_{n2,t} & \dots & \sigma_{nn,t} \end{bmatrix} \begin{bmatrix} dZ_{1,t} \\ dZ_{2,t} \\ \dots \\ dZ_{n,t} \end{bmatrix}, \quad (2)$$

where $\boldsymbol{\mu_{s,t}}$ is a $n \times 1$ vector of the instantaneous expected risk-premia (drifts), $\boldsymbol{\sigma_{s,t}}$ is an $n \times n$ lower triangular volatility matrix, and $\boldsymbol{dZ_t}$ is a column vector with n increments of independent standard Wiener processes. Given that both $\boldsymbol{\mu_{s,t}}$ and $\boldsymbol{\sigma_{s,t}}$ are time-varying and conditioned by the state variable Y_t, if we consider $Y_t = i,\ i \in R$, we obtain:

$$\boldsymbol{\mu_{s,t}} = \boldsymbol{\mu_{s,i}} = \begin{bmatrix} \boldsymbol{\mu}_{1,i} \\ \boldsymbol{\mu}_{2,i} \\ \dots \\ \boldsymbol{\mu}_{n,i} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\sigma_{s,t}} = \boldsymbol{\sigma_{s,i}} = \begin{bmatrix} \sigma_{11,i} & 0 & \dots & 0 \\ \sigma_{21,i} & \sigma_{22,i} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \sigma_{n1,i} & \sigma_{n2,i} & \dots & \sigma_{nn,i} \end{bmatrix}, \quad (3)$$

in which, $\boldsymbol{\mu}_{j,i}$ coefficients and $\boldsymbol{\sigma_{s,i}}$ matrices are constant for each $j = \{1, 2, \dots, n\}$. We underline that $\boldsymbol{\sigma_{s,i}}$ elements are defined as partial volatilities, e.g., $\sigma_{21,i}$ denotes partial volatility of asset 2 in relation to the first Wiener process ($dZ_{1,t}$) in regime i, and also that $\boldsymbol{\sigma_{s,i}}\boldsymbol{\sigma_{s,i}^T}$ represent the regime-dependent variance-covariance matrix. Still, as the drifts are regime-dependent and simultaneously time-dependent, it means that they can vary in time even if the regime remains unchanged. Such drifts are stored in an $n \times m$ drift matrix:

$$\boldsymbol{D_{s,t}} = \begin{bmatrix} \boldsymbol{\mu}_{1,1} & \boldsymbol{\mu}_{1,2} & \dots & \boldsymbol{\mu}_{1,m} \\ \boldsymbol{\mu}_{2,1} & \boldsymbol{\mu}_{2,2} & \dots & \boldsymbol{\mu}_{2,m} \\ \dots & \dots & \dots & \dots \\ \boldsymbol{\mu}_{n,1} & \boldsymbol{\mu}_{n,2} & \dots & \boldsymbol{\mu}_{n,m} \end{bmatrix}. \quad (4)$$

With such assets and state variable processes, we obtain the regime-parameters through the maximum likelihood (ML) estimation methodology, which are: the drift matrix $\boldsymbol{D}_{\boldsymbol{s},\boldsymbol{t}}$, the volatility matrix $\boldsymbol{\sigma}_{\boldsymbol{s},\boldsymbol{i}}$ and the transition probabilities $P_{ij,\Delta t}$ (where $j \neq i \in R$). The number of parameters to be estimated is $[mn + mn(n+1) \div 2 + m(m-1)]$. Given the unobservable nature of the regimes, according to Hamilton's (1989), we assume that investors can infer the regimes' occurrences through filtered probabilities observing the assets' past returns.

4.3.2 THE PORTFOLIO STRATEGY

Considering W_t as the wealth in t and $\boldsymbol{\alpha}_{\boldsymbol{t}}$ as the $1 \times n$ vector of portfolio shares of the risky assets, and $(1 - \boldsymbol{\alpha}_{\boldsymbol{t}}\mathbf{1})$ as the riskless asset share, wealth dynamics can be expressed as:

$$\begin{aligned} dW_t &= (1 - \boldsymbol{\alpha}_{\boldsymbol{t}}\mathbf{1})W_t\, r_f\, dt + W_t\, \boldsymbol{\alpha}_{\boldsymbol{t}} \frac{dS_t}{S_t} \\ &= W_t\, r_f\, dt + W_t\, \boldsymbol{\alpha}_{\boldsymbol{t}}[\boldsymbol{D}_{\boldsymbol{s},\boldsymbol{t}}\, \boldsymbol{\pi}_{\boldsymbol{t}}\, dt + (\boldsymbol{V}\boldsymbol{\pi}_{\boldsymbol{t}})\, \boldsymbol{dZ}_{\boldsymbol{t}}], \end{aligned} \tag{5}$$

where $\mathbf{1}$ is a column vector of n ones, $\frac{dS_t}{S_t}$ is the column vector with n infinitesimal risky asset returns, $\boldsymbol{\pi}_{\boldsymbol{t}}$ is a column vector with the m filtered probabilities in t and $\boldsymbol{V}$ is an $1 \times m$ row vector containing the regime-dependent covariance matrices ($\boldsymbol{\sigma}_{\boldsymbol{s},\boldsymbol{i}}$).

Utility function. In the CGL model, the investor's preferences are characterized as continuous-time and modeled by the stochastic utility function from Duffie and Epstein (1992):

$$J_t = E_t \left[\int_{u=t}^{T} f(C_u, J_u)\, du + \frac{W_T^{1-\gamma}}{1-\gamma} \right], \tag{6}$$

where E_t is the expected value in the current moment (t); T is the investment horizon; f is the recursive aggregator of the utility function J_t in function of consumption rate C_u (in moment u) and J_u, the continued utility in u. In turn, W_T is the investor's terminal wealth while γ is the risk aversion coefficient. The following function details the utility function aggregator:

$$f(C,J) = \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)\,J\left\{\left[\frac{C}{[(1-\gamma)\,J]^{\frac{1}{1-\gamma}}}\right]^{1-\frac{1}{\psi}} - 1\right\}, \tag{7}$$

where β is the time preference rate of the investor's utility (felicity) and ψ is the elasticity of intertemporal substitution, i.e., consumption choices over time. Thus, we must set ψ, β and γ to configure the strategy using the recursive utility.

Campani and Garcia (2019) analyze the sensitivity of consumption and portfolio choices in a single-state model. They indicate that ψ affects consumption preferences but barely affects the allocation strategy. Campani, Garcia, and Lewin (2021) find a similar conclusion for a regime-switching model. So, we can study the allocation strategy and disregard intermediary consumption, defining $\psi = \infty$. It represents the investor that waits for the terminal horizon to consume the wealth. In a problem without intermediary consumption, consequently, the value of β will not significantly affect the allocation strategy. Campani, Garcia, and Lewin (2021) and Guidolin and Timmermann (2007) also show that the investment horizon has a negligible impact on the strategy considering frequent rebalancing. And, as both, we consider $\gamma = 5$.

Campani, Garcia, and Lewin (2021) demonstrate that the general solution quantifying the investor total optimal utility in t ($V_t = \sup J_t$) admits the wealth-separable solution:

$$V(W_t, \boldsymbol{\pi_t}, \tau) = H(\boldsymbol{\pi_t}, \tau)\, \frac{W_t^{1-\gamma}}{1-\gamma}\,, \tag{8}$$

where $\tau = T - t$ is the time until the final horizon and $H(\boldsymbol{\pi_t}, \tau)$ is a function in terms of the time to horizon and the regimes' probabilities vector. But, as an exact analytical expression for $H(\boldsymbol{\pi_t}, \tau)$ is not yet available in the literature, Campani, Garcia, and Lewin (2021), based on the Bellman equation solve the problem with the following approximate analytical expression:

$$H(\boldsymbol{\pi_t}, \tau) = exp\left[A_0(\tau) + \sum_{i=1}^{m} A_i(\tau)\, \pi_{i,t} + \sum_{i=1}^{m} B_i(\tau)\pi_{i,t}^2 + \sum_{j<i} C_{ij}(\tau)\, \pi_{i,t}\, \pi_{j,t}\right], \tag{9}$$

where $\pi_{i,t}$ is regime i probability at time t. Meanwhile, $A_0, A_i, B_i,$ and C_{ij} are time-horizon coefficients obtained from the solution of the Bellman equation under a system of partial differential equations (PDE). Campani, Garcia, and Lewin (2021) demonstrate the PDE, and provide the details for solving the Bellman equation.

Portfolio weights. Given the approximate solution for $V(W_t, \boldsymbol{\pi_t}, \tau)$ and the coefficients $A_0, A_i, B_i,$ and C_{ij} from function $H(\boldsymbol{\pi_t}, \tau)$, the CGL model presents the optimal weights for the regime-switching allocation using the recursive utility function given by the following form:

$$\boldsymbol{\alpha_t} = \frac{1}{\gamma}\left(\boldsymbol{D_{s,t}}\,\boldsymbol{\pi_t}\right)^T[(\boldsymbol{V\pi_t})(\boldsymbol{V\pi_t})^T]^{-1} + \frac{1}{\gamma}\sum_{i=1}^{m}\left[A_i(\tau) + 2B_i(\tau)\,\pi_{i,t} + \sum_{j\neq i} C_{ij}(\tau)\,\pi_{j,t}\right]\boldsymbol{\sigma_{i,\pi}}\,(\boldsymbol{V\pi_t})^{-1}, \tag{10}$$

where $\boldsymbol{\alpha_t} = [\boldsymbol{\alpha}_{\mathbf{1},t} \;\; \cdots \;\; \boldsymbol{\alpha}_{\boldsymbol{n},t}]$, $\boldsymbol{\sigma_{i,\pi}} = \left[\sigma_{i1,\pi} \;\; \sigma_{i2,\pi} \;\; \dots \;\; \sigma_{in,\pi}\right]$, $i \in R$ and $j = \{1, 2, \dots, n\}$.

Maximum leverage control (MaxLev). Equation 10 indicates the optimal weights, but we must control them within the maximum leverage permitted by the investor. Hence, we apply Lewin and Campani's (2022) MaxLev to constrain leverage in the CGL model. First, we collect the unconstrained weights at 100% probability for each regime through equation (10) with $\boldsymbol{\hat{\pi}_i} = i^{th}$ column of an m-order identity matrix. Then, using a $1 \times m$ vector, we store the unconstrained leverages conditioned by each regime ($\boldsymbol{UncLev}$), whose elements are:

$$UncLev_i = \left[\sum_{k=1}^{n+1} \max\left(\boldsymbol{\hat{\alpha}}_{\boldsymbol{\hat{\pi}_i,k}}, 0\right)\right] - 1, \;\text{ with }\; i \in R = \{1, 2, \dots, m\}, \tag{11}$$

where $\boldsymbol{\hat{\alpha}}_{\boldsymbol{\hat{\pi}_i,k}}$ is the k^{th} element of the row vector $\boldsymbol{\hat{\alpha}}_{\boldsymbol{\hat{\pi}_i}} = \left[\boldsymbol{\alpha}_{\boldsymbol{\hat{\pi}_i}} \quad \left(1 - \boldsymbol{\alpha}_{\boldsymbol{\hat{\pi}_i}}\mathbf{1}\right)\right]$ formed by the risky asset weight vector and the riskless asset weight, all conditioned by $\boldsymbol{\hat{\pi}_i}$. We define the maximum leverage values ($L_{b,i}$) conditioned by regime, building a $z \times m$ matrix, where z is a positive integer representing the number of policies investigated at the research and $b = \{1, 2, \dots, z\}$:

$$\boldsymbol{MaxLev} = \begin{bmatrix} L_{11} & \cdots & L_{1m} \\ \vdots & \ddots & \vdots \\ L_{z1} & \cdots & L_{zm} \end{bmatrix}. \tag{12}$$

Then we compute the necessary adjustments for γ to confine $UncLev_i$ within the limits stated in $\boldsymbol{MaxLev}$, with which we infer the risk parameter ($\gamma = 5$) conditionally to the limits and regimes. The procedure emerges on a new $z \times m$ matrix whose elements are:

$$\gamma_{b,i} = \gamma \times max\,[\,(1 + UncLev_i) \div (1 + L_{b,i})\,,\ 1\,]\,, \tag{13}$$

where the maximum operator preserves $\gamma = 5$ when the element $UncLev_i$ is already below the limit imposed by $L_{b,i}$. Multiplying the new matrix b^{th} row (expressed by $\boldsymbol{\gamma_b} = [\gamma_{b,1} \quad \cdots \quad \gamma_{b,m}]$) by the column vector of regimes probabilities at time t ($\boldsymbol{\pi_t}$), we obtain a dynamic value as our risk parameter, which is an average of $\boldsymbol{\gamma_b}$ (dynamically) weighted by $\boldsymbol{\pi_t}$. Plugging it on equation (10), we then find the weights constrained by policy b at time t:

$$\begin{aligned}\boldsymbol{\alpha_{b,t}} = \frac{1}{(\boldsymbol{\gamma_b \pi_t})} \left(\boldsymbol{D}_{s,t}\,\boldsymbol{\pi_t}\right)^T [(\boldsymbol{V\pi_t})(\boldsymbol{V\pi_t})^T]^{-1} \\ + \frac{1}{\gamma}\textstyle\sum_{i=1}^{m}\left[A_i(\tau) + 2B_i(\tau)\,\pi_{i,t} + \sum_{j\neq i} C_{ij}(\tau)\,\pi_{j,t}\right] \boldsymbol{\sigma_{i,\pi}}\,(\boldsymbol{V\pi_t})^{-1}.\end{aligned} \tag{14}$$

4.3.3 THE FOUR-FACTOR MODEL

Data set. We calculated the risk factors using the stocks listed in IBrX 100, the index of the 100 most liquid Brazilian stocks. The B3 exchange rebalances the index compositions three times a year since 1996, always on the first Monday of January, May, and September. We discarded the first year and extracted the daily prices from 1997 to

2022 from Economatica (we converted the series to weekly data to apply the regime-switching model).

Based on rebalancing dates, we update the stock universe three times a year to keep up with new index compositions. In addition, we control the survival bias by considering the complete set of stocks listed in the index composition at any given time as our universe. Thus, when necessary, we have calculated the portfolios considering stocks that have been eventually delisted afterwards. Using the index naturally imposed a liquidity filter on the stocks' selection. On top of this, first, we kept only one stock per company (the most negotiated one). Second, we filtered out: penny stocks, stocks with a negative book value (to eliminate those high default risk), and stocks without trades in more than 20% of the daily observations in the year formed before the composition report (to mitigate potential distortions on MOM's calculation).

Construction. Following Fama and French (1993) and Cahart (1997), the four-factor model relies on multivariate sorts to build value-weight portfolios based on size (market equity), book-to-market (BM), and prior (2-12) returns (since the first factor is the market portfolio). We followed Fama (2017) to set the prior (2-12) configuration. It means, for example, that in January 2001, stocks were ranked according to prior 2-12 months total returns based on their continuously compounded returns from January 2000 to November 2000.

The cited authors create six bivariate portfolios. They classify the stocks in two portfolios ranked by size, using the median to separate big and small stocks. At the same time, they classify stocks into three portfolios ranked by the BM, using the 30th and 70th percentiles to separate high (value), neutral and low (growth) stocks. So, applying the same procedure using past returns instead of BM, they also get high (winners), neutral, and low (losers) stocks. Combining the sort by size with the sort by BM, they obtain six bivariate portfolios. Likewise, combining the sorts by size and prior returns, they obtain another six portfolios.

Although we also make bivariate portfolios, we construct nine portfolios instead of six. First, we classify the stocks in three portfolios ranked by size, using the 40th and 60th percentiles to separate them into big, neutral (medium), and small stocks; we adopt this same procedure to rank the stocks by BM and prior returns. The configuration with such percentiles means we shrank the neutral (univariate) portfolios relative to Fama and French (1993): the reason for so is that we have way less stocks in Brazil, so we want to disregard less stocks (as we explain later, the neutral portfolios will be disregarded). Next, combining the sort by size with the sort by BM, we obtain nine bivariate portfolios. Then, we obtain another set of nine portfolios by repeating the procedure with prior returns instead of BM. Note that five out of any set of nine portfolios are bivariate types, where "neutral" is at least one of the possible combinations.

From the nine possible combinations formed between size and BM (or between size and prior returns), we discard the five portfolios formed with “neutral” to mitigate the correlation between factors. Finally, we weekly repeat the portfolios sort, matching the data frequency further used by the CGL model.

SMB (small minus big). As indicated, when sorting size and BM, after discarding the five portfolios formed with “neutral”, we obtain four portfolios: small value, small growth, big value, and big growth. Then, with the daily returns of such portfolios, we obtain SMB as:

$$SMB = \frac{(Small\,Value + Small\,Growth)}{2} - \frac{(Big\,Value + Big\,Growth)}{2}. \quad (15)$$

HML (high minus low). The abovementioned portfolios, following Fama (2017), provide HML as:

$$HML = \frac{(Small\,Value + Big\,Value)}{2} - \frac{(Small\,Growth + Big\,Growth)}{2}. \tag{16}$$

MOM (momentum). The nine bivariate portfolios formed on size and prior (2-12) returns originate MOM, where analogously to the previous factors, we discard the five portfolios formed by the combinations with “neutral”. Then, also following Fama (2017), MOM is defined as follows:

$$MOM = \frac{(Small\,High + Big\,High)}{2} - \frac{(Small\,Low + Big\,Low)}{2}, \tag{17}$$

where small high, big high, small low, and big low are the series of daily returns originated by those by the bivariate portfolios sorted by size and prior (2-12) returns.

Market factor (Mkt-rf). The excess returns of the IBrX 100 index over the returns from the risk-free asset represent the market factor. Following Lewin and Campani (2020a), we consider as the risk-free asset (rf) the return of the Brazilian CDI rate.

4.3.4 THE APPLICATION OF THE CGL MODEL

We apply the CGL model to allocate $n = 4$ risky assets (Mkt-rf, SMB, HML, and MOM) along with the risk-free asset (CDI).

The out-of-sample exercise. We built a 20-year exercise organized in 60 windows of observations. All of them started on January 8th, 1997, but the number of observations increased at a 4-month step: this procedure guarantees that we estimate the model based on the richest possible sample. We (re)estimate the model every four months, at the end of each window (a higher frequency did not significantly change the estimations). The first window, ending on December 30th, 2002, has 313 observations; the last one ends on August 31st, 2022, with 1339 observations. Hence, the regime parameters were (re)estimated via maximum likelihood with the observations on each window and held during the four subsequent months. At every new week, we define the strategy from the filtered probabilities estimated in t for $t+1$. Then, replicating only the available information at the investment decision moment, we observe the out-of-sample returns. This exercise extends from January 8th, 2003, to December 19th, 2022, encompassing 1356 weekly observations.

Number of regimes. To define the number of regimes (m), we consider $m = 1$ as the single regime model presented in the results section and that $m = 4$ is the highest number of states often observed in the literature. In addition, Guidolin and Ono (2006) indicate a saturation ratio between the number of estimated parameters and the series length, with values above 17. In our application, $m \leq 4$ present it. Thus, we tested models with $m = 2, 3, 4$. Table 1 shows their information criteria (IC), presenting 3 out of the 60 (re)estimations obtained using windows 1, 30, and 60. The IC is relatively stable during the (re)estimations and, as Table 1 indicates, $m = 4$ dominates the other models. Therefore, we apply a model with four regimes.

Table 1 – Information criteria

Window	m	AIC	BIC	H-Q
Oldest				
	2	-5.039	-5.024	-5.075
	3	-5.067	-5.043	-5.124
	4	-5.085	-5.051	-5.167
Intermediate				
	2	-14.854	-14.826	-14.886
	3	-14.923	-14.878	-14.974
	4	-15.033	-14.970	-15.106
Most recent				
	2	-24.435	-24.401	-24.465
	3	-24.616	-24.562	-24.665
	4	-24.809	-24.732	-24.877

Notes. The table indicates the information criteria for the models with $n = 4$ risky assets under 2, 3, and 4 regimes. Its columns present Akaike (AIC), Bayes-Schwartz (BIC), and Hannan-Quinn (H-Q) for three windows from the out-of-sample exercise. The oldest window was estimated from 08/Jan/1997 to 30/Dec/2002, the intermediate, from 08/Jan/1997 to 26/Dec/2012, and the most recent, from 08/Jan/1997 to 19/Dec/2022. The entire out-of-sample exercise was performed with 60 windows.

Transaction Costs. We assume the investor rebalances the portfolio every (end of) week, bearing transaction costs. We follow Gârleanu and Pedersen (2013) and Nystrup, Boyd, Lindström, and Madsen (2019), who fixed transaction costs at 10 basis points (0.10%) for dynamic asset allocation strategies. The last authors additionally propose to account for holding costs, charged at the risk-free rate over the short sales. In our application, holding costs are naturally considered, since the investor borrows at the risk-free rate for short selling.

4.3.5 PORTFOLIO LEVERAGE

We apply Lewin and Campani's (2022) MaxLev procedure to constrain the CGL's optimal solution to present the portfolios according to leverage levels and to observe different investment profiles. The unleveraged strategy is CGL MaxLev 0%, and the leveraged ones are CGL MaxLev 50%, 100%, 150%, 200%, and 250%. Section 4.3 shows the CGL results obtained using these configurations and identifies those closest to the benchmarks' leverage in order to appropriately perform the exercise comparisons.

Nonetheless, equations 15 to 17 show that factors have intrinsic leverage as they long and short their underlying portfolios, generating the risk premia over the risk-free (CDI). Equation 10 presents the vector of optimal risky assets' weights, creating the factor-based strategy. Likewise, it is the vector containing the optimal weights for each factor without adding up to 100%, as it does not carry a risk-free weight ($1 - \boldsymbol{\alpha}_t \mathbf{1}$). Still, to compute the dynamically constrained weights (equation 14), we assume a summation of weights to 100% by considering that an investment of \$100 on SMB, for example, is investing \$100 on the risk-free, buying \$100 (long) of small stocks, and short selling \$100 on the big stocks portfolio.

Therefore, referring to an unleveraged strategy, i.e., CGL MaxLev 0%, indicates that we do not suggest any extra leverage besides the intrinsic factors' leverage (100%). In turn, if the CGL model recommends 50% leverage, the investor will be 150% leveraged, given the factors' intrinsic leverage.

4.3.6 BENCHMARKS

We assume that the investor is willing to diversify between the Brazilian risk factors. Thus, we compare the strategies from the CGL model with four active and passive strategies. The benchmarks also differentiate between unleveraged and leveraged strategies, as we will present the CGL portfolio varying the leverage configuration.

Equal-weights portfolio ($\mathbf{1/n}$). DeMiguel, Garlappi, and Uppal (2009) demonstrate that a $1/n$ portfolio outperforms several dynamic models based on optimal rules, despite being a naïve strategy. Thus, it represents a benchmark for passive (and unleveraged) strategy.

Tangency Portfolio (Tangency). The tangency portfolio is based on quadratic utility preferences used to build the efficient frontier. The point where the upward-sloping line is tangent to the frontier of risky assets corresponds to this portfolio, a portfolio of risky assets only. We present it as a benchmark for the active unleveraged strategy, i.e., CGL MaxLev 0%, as it maximizes the Sharpe ratio (over the risky portfolios on the efficient frontier).

Single regime model (SR). The SR model corresponds to the recursive utility preferences of an investor who does not account for a multiple-regime economy. We use it to assess the impact of regime-switching on overall performance. As the SR strategy relies on leverage, it is a benchmark for the active leveraged strategies, i.e., CGL MaxLev 50% to 200%.

Constrained single regime model (SR CONS). First, we constrain the latter strategy by recalculating the optimal weights proportionally to an unleveraged portfolio. Then, SR CONS becomes a benchmark for active unleveraged strategies, with which we assess the impact of regime-switching on CGL MaxLev 0%.

4.3.7 ROBUSTNESS CHECK

Following Fugazza, Guidolin, and Nicodano (2015) and Campani, Garcia, and Lewin (2021), we use the annualized certainty equivalent returns (CER) to compare and rank different strategies. The authors provide the derivation of the following expression used to compute CER:

$$CER_i(\gamma, t) \equiv \frac{F}{T}\left\{\frac{1}{W_t}\left[\frac{1}{K-T}\sum_{\tau=1}^{K-T}\left[W_{\tau+T}\left(\widehat{\omega}_{i,t}(\gamma, T)\right)\right]^{\frac{1}{1-\gamma}}\right]-1\right\}, \quad (18)$$

where F is the data frequency (52 weeks per year), T is the horizon (520 weeks), K is the number of out-of-sample returns, $\widehat{\omega}_{i,t}$ are the proportions of the wealth invested in asset i, and W_t is the initial wealth (set to 1). The following section presents the CER differences to compare two portfolios., It will also report the 95% bootstrap confidence intervals drawn from 1,000,000 samples with replacement, using the bias-corrected and accelerated percentile method due to non-normalities in the out-of-sample returns.

4.4 RESULTS

4.4.1 THE FOUR-FACTOR MODEL

In Table 2, panel A presents the factors' long-run mean, volatility, and correlation, as single-state parameters. It shows that MOM's expected excess returns are positive during 2003-2022, while SMB and HML's are not distant from zero, indicating that big and growth stocks (low BM) did not outperform the short sale of their peers, as winners did. As expected, we also see a negative correlation between SMB and HML, as they originate from size and BM portfolios. Given that BM is a ratio between book equity (BE) and market equity (size), as size is updated daily (resulting from price), but BE changes on a lower frequency, in the long run, HML usually has a constant numerator with a denominator varying with size. Then, the correlations from HML and SMB with Mkt$-rf$ also present inverted signals. In the next section, reading the regime parameters, we infer the intuition behind it.

Table 2 – Estimated parameters

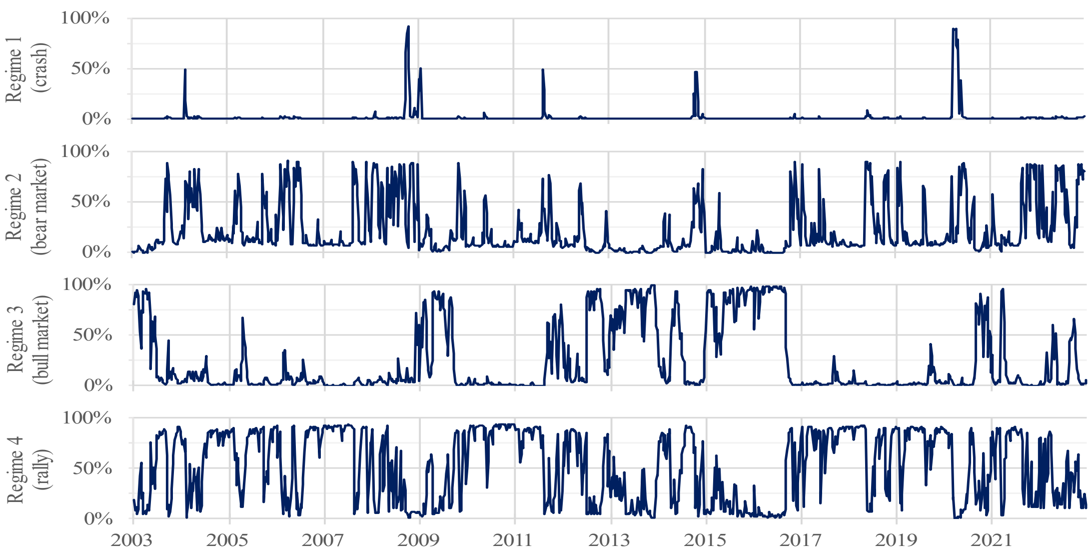
Panel A: Single State Model			**Mkt−rf**	**SMB**	**HML**	**MOM**
Expected returns						
			4.2%	-0.4%	0.0%	9,3%
Volatility and correlation matrix						
		Mkt−rf	27.9%			
		SMB	-.30	20.2%		
		HML	.30	-.42	18.0%	
		MOM	-.18	-.01	-.32	19,0%
Panel B.1: Four State Model			**Mkt−rf**	**SMB**	**HML**	**MOM**
Expected Returns	Regime 1 (crash)		-64,8%	-41.8%	-43.7%	61.3%
	Regime 2 (bear market)		-6,7%	5.3%	14.6%	-4.3%
	Regime 3 (bull market)		2,9%	-4.6%	-3.0%	9.6%
	Regime 4 (rally)		23,2%	3.8%	-2.1%	14.6%
Volatility and correlation matrix						
	Regime 1	Mkt−rf	86.6%			
	(crash)	SMB	-.59	63.9%		
		HML	.57	-.82	37.4%	
		MOM	-.15	.39	-.49	36,2%
	Regime 2	Mkt−rf	29.5%			
	(bear market)	SMB	-.12	22.4%		
		HML	.03	-.41	21.4%	
		MOM	-.09	-.32	.01	22,2%
	Regime 3	Mkt−rf	21.2%			
	(bull market)	SMB	-.12	14.7%		
		HML	.68	-.09	18.1%	
		MOM	-.66	.12	-.75	22,3%
	Regime 4	Mkt−rf	17.6%			
	(crash)	SMB	-.29	11.9%		
		HML	.16	-.35	12.0%	
		MOM	.12	.09	-.29	11,4%
Panel B.2: Four State Model			**Regime 1**	**Regime 2**	**Regime 3**	**Regime 4**
Transition Probabilities						
	Regime 1 (crash)		86,9%	12.9%	0.1%	0.1%
	Regime 2 (bear market)		1,6%	87.3%	1.1%	10.0%
	Regime 3 (bull market)		0,0%	0.0%	96.7%	3.3%
	Regime 4 (rally)		0,0%	7.0%	1.0%	92.0%
Ergodic Probabilities			3,6%	28.7%	22.8%	45.0%
Duration (weeks)			8	8	30	12

Notes. The table was computed with excess returns over the risk-free rate. Correlation matrices present the volatilities in their diagonals. Weekly returns and volatilities are annualized for presentation. The parameters were estimated with the complete data set. Market and rf returns are the IBrX 100, and the CDI rate returns, respectively. In turn, SMB, HML, and MOM are small minus big, high minus low, and momentum, respectively.

4.4.2 THE FOUR-REGIME MODEL

In Table 2, panels B.1 and B.2 show the parameters of the four economic states. The crash is the most pessimistic regime for Mkt-rf, SMB, and HML's returns, and their correlation is higher than in other regimes, suggesting that size and BM have a limited impact on crash returns. Contrastingly, the crash is the most optimistic regime for MOM: the estimated parameters show that crash and rally states generate momentum for stocks. Nevertheless, the ergodic probabilities and duration still reveal that crashes are rare and short, occurring only 3.58% of the time and lasting, on average, 8 weeks. Most of the time, the economy is in one of the three remaining states. Bear and bull markets are the least extreme regimes, but both usually shift to a rally, the most likely one 44.95% of the time.

Figure 1 – Out-of-sample probabilities



Notes. The out-of-sample probabilities are the filtered probabilities from $t + 1$ in t, estimated considering the data windows described in section 2.4.

Figure 1 shows that the bull market, despite the prolonged duration, is less recurrent than the bear market and the rally state. The last five years of the exercise illustrate the relevance of a regime-switching model for an active strategy in a factor-based portfolio. Despite the 2020 crash and the 2021 bull market, since 2017, the most frequent regimes are the bear market and rally. Although SMB's expected returns are approximately similar in the bear market and rally, they are not as impacting as HML and MOM's in each state. The returns from Mkt$-rf$, HML, and MOM change signals between those two states, and a passive investor or an active one who does not consider the regime-switching framework likely would not capture such state shifts.

Table 3 – Out-of-sample results

Strategies	**Return (%)**	**Vol. (%)**	**Skewness (%)**	**Kurtosis**	**MDD (%)**	**Leverage (%)**		
						5th Perc.	**Average**	**Maximum**
Unleveraged single-state models								
$1/n$	14.58	8.06	-108.96	7.02	16.06	0.00	0.00	0.00
SR CONS	14.80	9.99	-75.48	5.59	16.87	0.00	0.00	0.00
Leveraged single-state models								
Tangency	15.05	13.03	-106.38	9.70	32.06	0.00	6.08	55.33
SR	14.54	11.17	-80.34	4.54	20.22	0.00	12.51	37.76
Unleveraged regime-switching model								
CGL MaxLev 0%	15.65	6.33	-19.06	2.18	5.97	0.00	0.00	0.00
Leveraged regime-switching models								
CGL MaxLev 50%	17.76	10.55	-22.67	1.72	13.12	0.00	29.97	50.00
CGL MaxLev 100%	19.76	14.81	-18.97	2.30	19.25	4.59	65.11	100.00
CGL MaxLev 150%	21.56	18.74	-14.59	2.85	28.73	9.79	99.28	150.00
CGL MaxLev 200%	23.16	22.24	-20.14	2.96	37.84	15.46	131.71	200.00
CGL MaxLev 250%	24.28	25.25	-31.39	3.15	45.22	17.96	160.12	250.00
CGL Unconstrained	28.86	34.40	-37.99	4.52	55.05	25.40	244.02	716.84

Notes. The table presents the results from the out-of-sample exercise from 8/Jan/2003 to 19/Dec/2022. It shows annualized weekly average returns and volatilities (vol.). Returns are absolute, *i.e.*, not excess returns, and net of transaction costs. The table also indicates skewness, kurtosis (not excess kurtosis), and the maximum drawdown (MDD). Leverage is the sum of any strategy weights exceeding 100% on each observation; it does not account for the factors' intrinsic leverage. The right-most columns show the 5th percentile, average and maximum leverage. The minimum leverage is zero for the optimal CGL model; therefore, it is also zero on any CGL variation. CGL MaxLev 0% to 250% use the leverage constraining method introduced by Lewin and Campani (2022), while CGL unconstrained is the strategy without it. The SR and CGL strategies were optimized for $\gamma = 5$.

4.4.3 THE OUT-OF-SAMPLE PERFORMANCE

Table 3 presents the factor-based portfolios using single-state and regime-switching strategies. The $1/n$, as an equally weighted portfolio, is the passive strategy benchmark. Its volatility is lower than any other single-state active strategy in the exercise but higher than CGL MaxLev 0%, whose returns dominate the single states'. The higher-order statistical moments (skewness and kurtosis) also suggest that the CGL model offers lower risk than the benchmarks. At the same time, CGL MaxLev 0% had the only single-digit maximum drawdown (MDD).

Table 3 shows the leverage of single-state models ranges to 55.33%, making the CGL MaxLev 50% their most suitable comparable. Besides, their 5th percentile and average leverage are the same or the closest, while any maximum leverage set between 0% and 50% did not significantly affect the conclusions. On the other hand, CGL MaxLev 100% to 250% and CGL unconstrained illustrate different risk configurations where volatility is lower than or close to IBrX 100's volatility (table 2 shows that the market factor volatility is 27.95% p-a, which is practically the same in absolute returns). However, as the CGL MaxLev 100% to 250% does not have a similar leverage benchmark to be compared with, tables 4 and 5 suppress them for parsimonious.

Table 4 allows for comparing strategies in return-to-risk, i.e., by Sharpe ratio. It shows that in 2003-2022, the Sharpe ratios from CGL MaxLev 0% and 50% surpassed 62%, while the benchmarks only ranged to 43%. So, the CGL model outperformed the returns-to-risk benchmarks on the complete exercise. In addition, consistently presenting the lowest volatility, CGL MaxLev 0% offered a higher Sharpe ratio than both or at least one of the unleveraged benchmarks on each four-year interval. Meanwhile, CGL MaxLev 0% and 50% are the only strategies without any negative Sharpe ratio. These results add that the CGL model is a competitive strategy to diversify factor-based portfolios also in shorter investment horizons.

Table 4 – Sharpe ratio (%)

Panel A	**2003-2022**			**2003-2006**			**2007-2010**		
	Return	**Vol.**	**Sharpe**	**Return**	**Vol.**	**Sharpe**	**Return**	**Vol.**	**Sharpe**
Unleveraged single-state models									
$1/n$	14.58	8.06	42.90	26.18	7.49	105.03	15.30	7.63	57.70
SR CONS	14.80	9.99	36.82	24.80	10.93	59.31	11.13	9.29	2.45
Leveraged single-state models									
Tangency	15.05	13.03	30.14	25.12	14.27	47.69	11.26	9.54	3.76
SR	14.54	11.17	30.58	23.53	12.71	41.06	10.54	11.32	-3.14
Unleveraged regime-switching model									
CGL MaxLev 0%	15.65	6.33	71.58	26.86	7.18	119.00	15.83	6.40	77.07
Leveraged regime-switching model									
CGL MaxLev 50%	17.76	10.55	62.92	28.44	11.65	86.91	17.85	10.73	64.81
Panel B	**2011-2014**			**2015-2018**			**2019-2022**		
	Return	**Vol.**	**Sharpe**	**Return**	**Vol.**	**Sharpe**	**Return**	**Vol.**	**Sharpe**
Unleveraged single-state models									
$1/n$	5.53	6.90	-60.00	15.27	8.14	54.36	11.62	9.75	55.42
SR CONS	12.88	7.37	43.57	9.37	9.96	-14.88	16.48	11.83	86.74
Leveraged single-state models									
Tangency	14.40	8.47	55.85	7.38	17.87	-19.38	17.87	12.82	90.94
SR UNC	14.31	8.27	56.10	8.47	10.81	-21.95	16.43	12.22	83.54
Unleveraged regime-switching model									
CGL MaxLev 0%	11.20	5.94	25.86	14.22	5.97	56.51	10.85	5.91	78.24
Leveraged regime-switching model									
CGL MaxLev 50%	14.95	10.21	51.74	15.70	10.29	47.17	12.50	9.80	64.07

Notes. The table presents the Sharpe ratio of the out-of-sample exercise (2003-2022); and in four-year intervals. It shows annualized weekly average returns and volatilities (vol.). Returns are absolute, *i.e.*, not excess returns, and net of transaction costs. CGL MaxLev 0% and 50% use the leverage constraining method introduced by Lewin and Campani (2022). We suppressed the CGL model with higher leverage as they do not have a suitable comparable in terms of leverage. The SR and CGL strategies were optimized for $\gamma = 5$.

Table 5 indicates the differences between annualized certainty equivalent returns (ΔCER) and their confidence intervals. With positive ΔCER and positive confidence intervals, the first line demonstrates that CGL MaxLev 0% outperforms $1/n$ and SR

CONS with statistical significance. Analogously, the second line offers evidence on the leveraged portfolios comparison: CGL MaxLev 50% statistically outperforms the Tangency and SR portfolios.

Table 5 – Certainty equivalent returns differences (ΔCER, %)

Regime-switching model	Single-state models			
	$\mathbf{1/n}$	**SR CONS**	**Tangency**	**SR**
CGL MaxLev 0%	1.30 [0.90 - 2.19]	2.09 [0.43 - 3.84]	1.90 [-1.01 - 2.93]	5.30 [-4.54 - 8.86]
CGL MaxLev 50%	3.82 [0.86 - 6.14]	4.22 [-1.75 - 6.71]	2.22 [0.27 - 3.49]	5.82 [1.14 - 7.78]

Notes. The table indicates the differences between annualized certainty equivalent returns (ΔCER) computed with absolute returns net of transaction costs. The differences correspond to the CER from the models indicated in the horizontal panels, minus the CER from the benchmarks in the columns – computed according to section 2.7. The horizontal panels present CGL MaxLev 0% and 50%, as we suppressed the CGL model with higher leverage as they do not have a suitable comparable in terms of leverage. The SR and CGL strategies were optimized for $\gamma = 5$. Below the ΔCER, we report the 95% bootstrap confidence intervals drawn from 1,000,000 samples with replacement, with the bias-corrected and accelerated percentile method. The out-of-sample exercise was conducted with weekly observations from 08/Jan/2003 to 19/Dec/2022.

4.5 CONCLUSION

Using regime-switching models to diversify factor-based portfolios show promising results, according to Perez-Quiros and Timmermann (2000), Black and McMillan (2004), Guidolin and Timmermann (2008b), Tu (2010), Gulen et al. (2011), and Chincoli and Guidolin (2017). We reinforced this conclusion under an expended setting. In developed markets, the literature presents it under power utility. In emerging markets, it presents it under a still more simplistic framework, with quadratic utility preferences. Therefore, we propose to close the gap in the factor-based investing literature using a more realistic utility function: the stochastic recursive utility. We also concentrate on a single emerging market, Brazil, due to its regional relevance and to avoid the impact of exchange rates over the estimations.

We built an out-of-sample exercise to compare the CGL model with passive and active diversification strategies as benchmarks. As the optimal solution of regime-switching strategies often indicates high leverage, we used Lewin and Campani's (2022) MaxLev procedure to constrain the weights of the CGL portfolio in order to keep the strategy feasible for practical matters. The procedure was even more necessary to avoid overleveraging, given that factors are intrinsically leveraged, as they long and short their underlying stock portfolios. As a result, we used CGL MaxLev 0% to compare with the unleveraged benchmarks, and CGL MaxLev 50% to compare with the leveraged benchmarks., Section 4.3 shows that the 5th percentile, median, and maximum leverage from CGL MaxLev's 50% are the closest to the leveraged benchmarks. Thus, it represented the most suitable CGL portfolio to compare with them, while maximum leverage set between 0% and 50% did not significantly affect our conclusions.

The Sharpe ratios indicate that both, leveraged and unleveraged CGL strategies, outperform the benchmarks in the whole sample of the exercise. In shorter subsamples, first, we observe that CGL MaxLev 0% consistently presents the lowest volatility while offering a Sharpe ratio higher than at least one of the unleveraged benchmarks. Second, differently from any benchmarks, the Sharpe ratios of leveraged and unleveraged CGL strategies are positive in every subsample investigated. Finally, as a robustness check, the certainty equivalent returns reveal that both CGL strategies outperform their benchmarks with statistical significance.

The current research showed that regime-switching models provide competitive strategies to efficiently diversify factor-based portfolios in the Brazilian stock market. Further studies can address more factors and other markets. Another path for further investigation is using a higher data frequency to update the factors more frequently. For example, rapid updates to sort small, neutral, and big stocks might generate an even better momentum strategy. We highlight that sophisticated investors and fund managers may apply the methodology used in this article to provide advanced strategies based on risk factors that can benefit from active management. Doing so, they will be likely to outperform all or at least most of the factors in the long run.

4.6 REFERENCES

Ang, Andrew (2014). Asset Management: A Systematic Approach to Factor Investing. Oxford University Press, 2014.

Asness, Clifford; Moskowitz, Tobias; and Pedersen, Lesse (2013). Value and Momentum Everywhere. *Journal of Finance*, 68(3), 929–985. https://doi.org/10.1111/jofi.12021.

Barroso, Pedro and Santa-Clara, Pedro (2015). Momentum Has Its Moments. *Journal of Financial Economics*, 116(1), 111–120. https://doi.org/10.1016/j.jfineco.2014.11.010.

Black, Angela and McMillan, David (2004). Non-linear Predictability of Value and Growth Stocks and Economic Activity. *Journal of Business Finance & Accounting*, 31, 439–474. https://doi.org/10.1111/j.0306-686X.2004.00546.x.

Blitz, David (2020). Factor Performance 2010–2019: A Lost Decade? *The Journal of Beta Investment Strategies*, 11(2), 57–65. https://doi.org/https://doi.org/10.3905/jii.2020.1.090.

Campani, Carlos Heitor and Garcia, René (2019). Approximate Analytical Solutions for Consumption/Investment Problems Under Recursive Utility and Finite Horizon. *The North American Journal of Economics and Finance*, *48*, 364–384. https://doi.org/10.1016/j.najef.2019.03.005.

Campani, Carlos Heitor; Garcia, René; and Lewin, Marcelo (2021). Optimal Portfolio Strategies in the Presence of Regimes in Asset Returns. *Journal of Banking and Finance*, *123*(106030), 1–17. https://doi.org/10.1016/j.jbankfin.2020.106030.

Carhart, Mark (1997). On Persistence in Mutual Fund Performance. *Journal of Finance*, 52(1), 57–82. https://doi.org/10.1111/j.1540-6261.1997.tb03808.x.

Chague, Fernando and Bueno, Rodrigo (2008). The CAPM and Fama-French Models in Brazil. *Latin American Meeting of the Econometric Society*, working paper.

Chen, Jieting and Kawaguchi, Yuichiro (2018). Multi-Factor Asset-Pricing Models Under Markov Regime Switches: Evidence From The Chinese Stock Market. *International Journal of Financial Studies*, *6*(2). https://doi.org/10.3390/ijfs6020054.

Chincoli, Francesco and Guidolin, Massimo (2017). Linear and Nonlinear Predictability in Investment Style Factors: Multivariate Evidence. *Journal of Asset Management*, 18(6), 476–509. https://doi.org/10.1057/s41260-017-0048-5.

Daniel, Kent and Moskowitz, Tobias (2016). Momentum Crashes. *Journal of Financial Economics*, 122(2), 221–247. https://doi.org/10.1016/j.jfineco.2015.12.002.

DeMiguel, Victor; Garlappi, Lorenzo; and Uppal, Raman (2009). Optimal Versus Naive Diversification: How Inefficient is the 1/N Portfolio Strategy? *The Review of Financial Studies*, *22*(5), 1915–1953. https://doi.org/10.1093/rfs/hhm075.

Duffie, Derrel and Epstein, Larry (1992). Stochastic Differential Utility. *The Econometric Society*, *60*(2), 353–394. https://doi.org/10.2307/2951600.

Fama, Eugene (2017). The Fama Portfolio: Selected Papers of Eugene F. Fama. University of Chicago Press. https://doi.org/10.7208/chicago/9780226426983.001.0001.

Fama, Eugene and French, Kenneth (1993) Common Risk Factors in the Returns on Stocks and Bonds. *Journal of Financial Economics*, 33(1), 3–56. https://doi.org/10.1016/0304-405X(93)90023-5.

Fama, Eugene and French, Kenneth (2015). Dissecting Anomalies with a Five-Factor Model. *Review of Financial Studies*, 29(1), 69–103. https://doi.org/10.1093/rfs/hhv043.

Ferson, Wayne; Siegel, Andrew; and Xu, Pisun (2006). Mimicking Portfolios with Conditioning Information. *Journal of Financial and Quantitative Analysis*, 41(3), 607–635. https://doi.org/10.1017/S0022109000002556.

Fugazza, Carolina; Guidolin, Massimo; and Nicodano, Giovanna (2015). Equally Weighted vs. Long-Run Optimal Portfolios. *European Financial Management*, *21*(4), 742–789. https://doi.org/10.1111/eufm.12042.

Gârleanu, Nicolae and Pedersen, Lasse (2013). Dynamic Trading with Predictable Returns and Transaction Costs. *The Journal of Finance*, *68*(6), 2309–2340. https://doi.org/10.1111/jofi.12080.

Guidolin, Massimo and Hyde, Stuart (2012). Can VAR Models Capture Regime Shifts in Asset Returns? A Long-Horizon Strategic Asset Allocation Perspective. *Journal of Banking and Finance*, *36*(3), 695–716. https://doi.org/10.1016/j.jbankfin.2011.10.011.

Guidolin, Massimo and Ono, Sadayuki (2006). Are the Dynamic Linkages Between the Macroeconomy and Asset Prices Time-Varying? *Journal of Economics and Business*, *58*(5–6), 480–518. https://doi.org/10.1016/j.jeconbus.2006.06.009.

Guidolin, Massimo and Timmermann, Allan (2007). Asset Allocation Under Multivariate Regime Switching. *Journal of Economic Dynamics and Control*, *31*(11), 3503–3544. https://doi.org/10.1016/j.jedc.2006.12.004.

Guidolin, Massimo and Timmermann, Allan (2008a). International Asset Allocation Under Regime Switching, Skew, and Kurtosis Preferences. *The Review of Financial Studies* 21(2), 889-935. https://doi.org/10.1093/rfs/hhn006.

Guidolin, Massimo and Timmermann, Allan (2008b). Size and Value Anomalies Under Regime Shifts. *Journal of Financial Econometrics*, 6(1), 1–48. https://doi.org/10.1093/jjfinec/nbm021.

Gulen, Huseyin; Xing, Yuhang; and Zhang, Lu (2011). Value Versus Growth: Time-Varying Expected Stock Returns. *Financial Management*, 40(2), 381–407. https://doi.org/10.1111/j.1755-053X.2011.01146.x.

Haghani, Victor and Dewey, Richard (2016). A Case Study for Using Value and Momentum at the Asset Class Level. *Journal of Portfolio Management*, 42(3), 101–113. https://doi.org/10.3905/jpm.2016.42.3.101.

Hamilton, James. D. (1989). A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle. *The Econometric Society*, *57*(2), 357–384. https://doi.org/10.2307/1912559.

Honda, Toshiki (2003). Optimal Portfolio Choice for Unobservable and Regime-Switching Mean Returns. *Journal of Economic Dynamics and Control*, 28 (1), 45–78. https://doi.org/10.1016/S0165-1889(02)00106-9.

Hou, Kewei; Xue, Chen; and Zhang, Lu (2015). Digesting Anomalies: an Investment Approach. *The Review of Financial Studies*, 28(3), 650–705. https://doi.org/10.1093/rfs/hhu068.

Lewin, Marcelo and Campani, Carlos Heitor (2020a). Portfolio Management Under Multiple Regimes: Strategies that Outperform the Market. *Revista de Administração Contemporânea*, *24*(4), 300–316. https://doi.org/10.1590/1982-7849rac2020190161.

Lewin, Marcelo and Campani, Carlos Heitor (2020b). Portfolio Management Under Multiple Regimes: Out-of-sample Performance. *Brazilian Review of Finance*, *18*(3), 52. https://doi.org/10.12660/rbfin.v18n3.2020.81210.

Lewin, Marcelo and Campani, Carlos Heitor (2022). Constrained Portfolio Strategies in a Regime-Switching Economy. *Financial Markets and Portfolio Management*, forthcoming. https://doi.org/10.1007/s11408-022-00414-x.

Lintner, John (1965). The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets. *Review of Economics and Statistics*, 47, 13–37. https://doi.org/10.1016/B978-0-12-780850-5.50018-6.

Liu, Hening (2011). Dynamic Portfolio Choice Under Ambiguity and Regime Switching Mean Returns. *Journal of Economic Dynamics and Control*, 35(4), 623–640. https://doi.org/10.1016/j.jedc.2010.12.012.

Nystrup, Peter; Boyd, Stephen; Lindström, Erik; and Madsen, Henrik (2019). Multi-Period Portfolio Selection with Drawdown Control. *Annals of Operations Research*, *282*(1–2), 245–271. https://doi.org/10.1007/s10479-018-2947-3.

Perez-Quiros, Gabriel and Timmermann, Allan (2000) Firm Size and Cyclical Variations in Stock Returns. *Journal of Finance*, 55(3), 1229–1262. https://doi.org/10.1111/0022-1082.00246.

Rouwenhorst, Geert (1999). Local Return Factors and Turnover in Emerging Stock Markets. *Journal of Finance*, 54(4), 1439–1464. https://doi.org/10.1111/0022-1082.00151

Santos, José; Famá, Rubens, and Mussa, Adriano (2012). A Adição do Fator de Risco Momento ao Modelo de Precificação de Ativos dos Três Fatores de Fama & French Aplicado ao Mercado Acionário Brasileiro. *REGE-Revista de Gestão*, 19(3), 453–472. https://doi.org/10.5700/rege473.

Sass, Jörn and Haussmann, Ulrich (2004). Optimizing the Terminal Wealth Under Partial Information: The Drift Process as a Continuous Time Markov Chain. *Finance and Stochastics*, 8(4), 553–577. https://doi.org/10.1007/s00780-004-0132-9.

Sharpe, William (1964). Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk. *Journal of Finance*, 19(3) 425–442. https://doi.org/10.1111/j.1540-6261.1964.tb02865.x.

Tu, Jun (2010). Is Regime Switching in Stock Returns Important in Portfolio Secisions? *Management Science*, 56(7), 1198–1215. https://doi.org/10.1287/mnsc.1100.1181.

Wachter, Jessica (2002). Portfolio and Consumption Decisions under Mean-Reverting Returns: An Exact Solution for Complete Markets. *Journal of Financial and Quantitative Analysis*, 37(1), 63–91. https://doi.org/10.2307/3594995.

5. FINAL CONSIDERATIONS

The Thesis aimed to improve the allocation strategy presented by Campani, Garcia, and Lewin (2021), *i.e.*, the CGL model, so that professional investors can apply it in practice. Thus, in the three articles that comprise the Thesis, with out-of-sample exercises, we evaluate the model's performance and robustness in different markets, asset classes, and constraining policies.

The first article, exploring a Brazilian portfolio of stocks and bonds, concluded that the CGL optimal (unconstrained) returns outperformed all benchmarks in six out of ten years of the exercise, while the CGL's average returns statistically outperformed theirs.

The second article investigates a single asset class portfolio formed with US equities. Although relative to the previous paper, it enabled studying a much larger sample (57 years), CGL results remained competitive. Furthermore, the second article introduced a leverage-constraining method and a rebalancing filter to constrain CGL optimal strategies. Thus, we identified constrained CGL portfolios (i) with lower volatility and higher Sharpe ratios than constrained benchmarks and (ii) outperforming unconstrained regime-switching models with statistical significance.

The last article identified Fama and French (1993) and Cahart's (1997) four-factors model in Brazilian stocks: SMB (small minus big), HML (high minus low), and MOM (momentum), in addition to the market factor. Then, we integrated such a model into the CGL model and compared it to passive and active investment strategies in an out-of-sample exercise, net of costs. The CGL model's Sharpe ratio outperforms the competitors' in the complete out-of-sample exercise and in shorter subsamples. This article also reveals that the CGL portfolio without any leverage offers the lowest volatility relative to the benchmarks (but with competitive returns). Finally, the certainty equivalent returns show that the CGL model outperforms competitors with statistical significance.

The three articles consistently indicate that, in the long run, the CGL model is competitive relative to benchmarks. As the scope of our study was to demonstrate that such a framework is feasible for professional investors, ultimately, the Thesis presents the building blocks for implementing portfolio strategies in the presence of multiple regimes using recursive utility function for optimization.

Future studies shall find research opportunities investigating different portfolios, such as commodities and other risk factors. New investigation paths can also address more complex cost structures like dynamically observed illiquidity costs or creating methods to control drawdowns with stop-loss triggers. Despite research being an (almost) unlimited path, sophisticated investors and fund managers may adopt the methodology employed in the Thesis, the CGL model, to benefit from advanced allocation strategies.

6. REFERENCES

Campani, Carlos Heitor; Garcia, René; and Lewin, Marcelo (2021). Optimal Portfolio Strategies in the Presence of Regimes in Asset Returns. *Journal of Banking and Finance*, *123*(106030), 1–17. https://doi.org/10.1016/j.jbankfin.2020.106030.

Carhart, Mark (1997). On Persistence in Mutual Fund Performance. *Journal of Finance*, 52(1), 57–82. https://doi.org/10.1111/j.1540-6261.1997.tb03808.x.

Duffie, Derrel and Epstein, Larry (1992). Stochastic Differential Utility. *The Econometric Society*, *60*(2), 353–394. https://doi.org/10.2307/2951600.

Epstein, Larr, and Zin, Stanley (1989). Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework. *Econometrica*, *57*(4), 937–969. https://doi.org/10.1142/9789814417358_0012.

Fama, Eugene and French, Kenneth (1993) Common Risk Factors in the Returns on Stocks and Bonds. *Journal of Financial Economics*, 33(1), 3–56. https://doi.org/10.1016/0304-405X(93)90023-5.

Munk, Claus (2013). Financial Asset Pricing Theory. *Oxford University Press*, 6th ed.. https://doi.org/10.1093/acprof:oso/9780199585496.001.0001.

Tu, Jun (2010). Is Regime Switching in Stock Returns Important in Portfolio Secisions? *Management Science*, 56(7), 1198–1215. https://doi.org/10.1287/mnsc.1100.1181.